Overview

- Complex mappings
- Conformal mappings
- Zhukovsky (Joukowsky) and Karman-Trefftz mappings

- $w = f(z)$ maps from (x, y) to (u, v). Examples:
  - $f(z) = z + z_0$ is a translation by z_0
  - $f(z) = e^{i\theta} z$ is a rotation by angle θ
  - $f(z) = e^z$ maps the strip $x \in \mathbb{R}$, $0 \leqslant y \leqslant \pi$
  - $f(z) = 1/z$ maps lines parallel to x, y axes onto circles
  - $f(z) = z^{1/n}$ maps the upper half plane onto a wedge of angle π/n
- Mappings can be composed, e.g. $h(z) = (g \circ f)(z) = g(f(z))$
  - $f(z) = e^{i\theta} z$, $g(z) = z + z_0$ is a rotation followed by a translation

- A mapping is said to be *conformal* if it preserves angles between curves
- Complex mappings $w = u + iv = f(z) = f(x + iy)$ where f is analytic are conformal

Theorem. *Conformal mappings* $f: D \to D'$, $D, D' \subseteq \mathbb{C}$, *preserve harmonic functions, i.e., if* G *is harmonic in* D', $G_{uu} + G_{vv} = 0$, *then* $g(x, y) = (U \circ f)(z)$ *is harmonic,* $g_{xx} + g_{yy} = 0$.

- Knowledge of a harmonic function in some domain can be used to find a harmonic function in a mapped domain (See Lesson25.nb)

- Conformal maps can be found between simple domains (e.g., a circle or the upper half plane) and shapes of practical interest, such as airfoils
- The Joukowski transform maps a circle onto air and hydrofoil shapes

$$z = Z + \frac{1}{Z}$$

  - $Z = e^{i\Theta} \Rightarrow z = e^{i\Theta} + e^{-i\Theta} = 2\cos\Theta$, e.g., a flate plate
  - $Z = e^{i\Theta} + i\delta, \delta \in (0, 1/2)$ a cambered, thick airfoil, symmetric fore-aft
  - $Z = e^{i\Theta} + \rho e^{i\phi}, \rho \in (0.2, 0.4)$, $\phi \in (-\pi/2, -\pi/4)$ cambered, thick, unsymmetrical airfoils, used in aircraft design 1920's

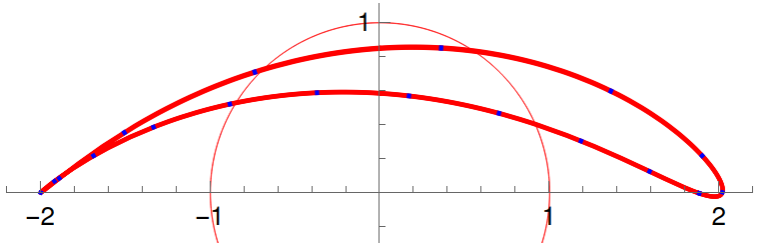

Figure 1. $\rho = 0.3, \phi = -1.38$ airfoil from Joukowski map

- In the 1930's better experimental arodynamics led to study of airfoils from

$$z = nb\frac{(Z+b)^n + (Z-b)^n}{(Z+b)^n - (Z-b)^n},$$

known as the Karman-Trefftz transform

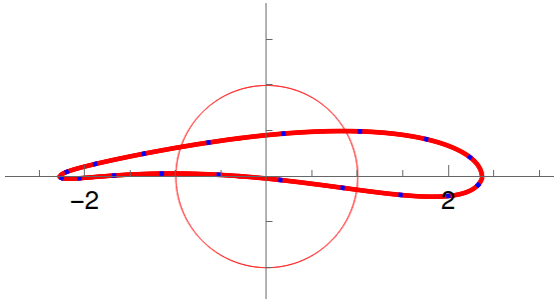

Figure 2. Karman-Trefftz airfoil with $b = 1.18, n = 1.92$, the image of circle $Z = e^{i\Theta} - 0.14e^{-0.785i}$.