Mid-term Examination

Solve the following problems (3 course points each). Present a brief motivation of your method of solution.

Parabolic coordinates (a,b) are related to rectangular coordinates (x,y) through

$$x = r\,s,\; y = \frac{1}{2}(s^2 - r^2),$$

have metric coefficients $L_r = L_s = \sqrt{r^2+s^2}$, and the Laplacian

$$\nabla^2 u = \frac{1}{r^2+s^2}(u_{rr}+u_{ss}),\; u_{rr} = \frac{\partial^2 u}{\partial r^2},\; u_{ss} = \frac{\partial^2 u}{\partial s^2}.$$

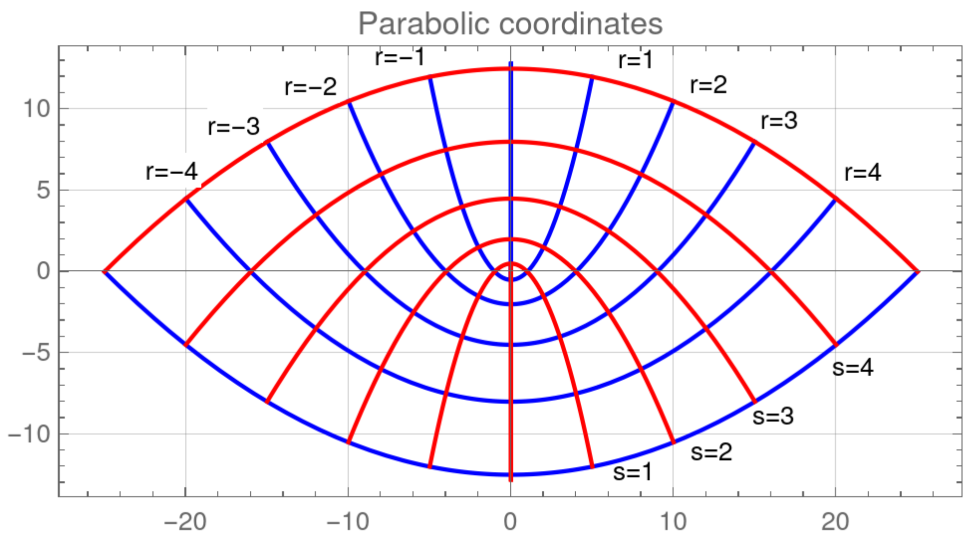

Figure 1. Parabolic coordinate lines of constant r (blue) and constant s (red). Parabolic coordinates are useful in optics and acoustics lens design.

1. Use separation of variables to find the parabolic coordinate solution $u(r,s,t)$ of the initial boundary value problem

$$\begin{array}{llll} u_{tt} = c^2\,\nabla^2 u, & t>0, & -5<r<5, & -5<s<5,\\ u(r,5,t)=0, & t>0, & -5<r<5, & \\ u(5,s,t)=0, & t>0, & & -5<s<5,\\ u(r,s,0)=f(r,s), & & -5<r<5, & -5<s<5,\\ u_t(r,s,0)=0, & & -5<r<5, & -5<s<5. \end{array}$$

2. Use separation of variables to find the parabolic coordinate solution $u(r,s,t)$ of the initial boundary value problem

$$\begin{array}{llll} u_t = c^2\,\nabla^2 u, & t>0, & -5<r<5, & -5<s<5,\\ u(r,5,t)=1, & t>0, & -5<r<5, & \\ u(5,s,t)=0, & t>0, & & -5<s<5,\\ u(r,s,0)=0, & & -5<r<5, & -5<s<5. \end{array}$$

3. Find the solution of Problem 1 through the Laplace transform

$$U(r,s,\tau) = \mathcal{L}\{u(r,s,t)\} = \int_0^\infty u(r,s,t)\,e^{-t\tau}\,\mathrm{d}t.$$

4. Find the solution of Problem 2 through the Fourier transform

$$U(a,s,t) = \mathcal{F}\{u(r,s,t)\} = \int_{-\infty}^{\infty} u(r,s,t)\, e^{iar}\, \mathrm{d}r.$$

Recall the Fourier transform of monomials formula

$$\mathcal{F}\{r^n\} = \int_{-\infty}^{\infty} r^n\, e^{-iar}\, \mathrm{d}r = 2\pi\, (-i)^n\, \delta^{(n)}(a),$$

where $\delta^{(n)}(a)$ is the n^{th} derivative of the Dirac delta function $\delta(a)$.