Mid-term Examination

Solve the following problems (4 course points each). Present a brief motivation of your method of solution.

1. Find the eigenvalues and eigenfunctions of the boundary value problem

$$x^2 y'' + x y' + 9\lambda y = 0,\, y'(1) = 0,\, y(e) = 0.$$

Solution. This is a Cauchy-Euler ODE. Trial solution of the form $y = x^r$ leads to

$$[r(r-1) + r + 9\lambda]x^r = 0,$$

and non-trivial solutions are obtained from roots of quadratic

$$r^2 + 9\lambda = 0 \Rightarrow r_{1,2} = \pm 3i\sqrt{\lambda},$$

leading to the general solution

$$y(x) = c_1 x^{3i\sqrt{\lambda}} + c_2 x^{-3i\sqrt{\lambda}},\, y'(x) = 3i\sqrt{\lambda}(c_1 x^{3i\sqrt{\lambda}-1} - c_2 x^{-3i\sqrt{\lambda}-1})$$

The boundary condition $y'(1) = 0$ implies $c_1 = c_2$, and $y(e) = 0$ leads to

$$c_1(e^{3i\sqrt{\lambda}} + e^{-3i\sqrt{\lambda}}) = 2c_1 \cos(3\sqrt{\lambda}) = 0.$$

Non-null solutions are obtained for $\cos(3\sqrt{\lambda}) = 0$, leading to eigenvalues

$$3\sqrt{\lambda_k} = k\pi + \frac{\pi}{2},\, k = 0, 1, 2, \ldots$$

with associated eigenfunctions

$$y_k(x) = x^{3i\sqrt{\lambda_k}} + x^{-3i\sqrt{\lambda_k}} = x^{i\left(k\pi + \frac{\pi}{2}\right)} + x^{-i\left(k\pi + \frac{\pi}{2}\right)}.$$

Since $x^a = e^{a\ln x}$, obtain

$$y_k(x) = e^{i\left(k\pi + \frac{\pi}{2}\right)\ln x} + e^{-i\left(k\pi + \frac{\pi}{2}\right)\ln x} = 2\cos\left[\left(k\pi + \frac{\pi}{2}\right)\ln x\right].$$

Since eigenfunctions are undetermined up to a multiplicative constant, the eigenvalue, eigenfunction pairs are

$$\lambda_k = \frac{1}{9}\left(k\pi + \frac{\pi}{2}\right)^2,\, y_k(x) = \cos\left[\left(k\pi + \frac{\pi}{2}\right)\ln x\right],\, k \in \mathbb{N}.$$

2. Use separation of variables to find the solution of

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + 2\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = 0.$$

Solution. Replacing $u(x, y) = X(x)\, Y(y)$ leads to

$$\frac{X''}{X} + 2\frac{X'}{X} = -\left(\frac{Y''}{Y} + 2\frac{Y'}{Y}\right) = \lambda,$$

and the constant-coefficient ODEs

$$X'' + 2X' - \lambda X = 0,\, Y'' + 2Y' + \lambda Y = 0,$$

with characteristic equations obtained from trial solutions $X(x) = e^{rx}$, $Y(y) = e^{sy}$

$$r^2 + 2r - \lambda = 0,\, s^2 + 2s + \lambda = 0.$$

The roots of the characteristic equation are

$$r_{1,2} = -1 \pm \sqrt{1+\lambda},\, s_{1,2} = -1 \pm \sqrt{1-\lambda}.$$

The general solution of the homogeneous PDE is therefore

$$u(x,y) = e^{-x-y}[c_1 e^{x\sqrt{1+\lambda}} + c_2 e^{-x\sqrt{1+\lambda}}][c_3 e^{y\sqrt{1-\lambda}} + c_4\, e^{-y\sqrt{1-\lambda}}].$$

Further analysis requires specification of boundary conditions.

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3. Solve the boundary-value problem

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, 0 < x < \pi, t > 0,$$

$$u(0,t) = 0, u(\pi,t) = 0, t > 0,$$

$$u(x,0) = \sin x, 0 < x < \pi.$$

Solution. Separation of variables $u(x,t) = X(x)T(t)$ leads to

$$\frac{X''}{X} = \frac{T'}{T} = -k^2,$$

where the constant $-k^2$ has been chosen negative to avoid non-physical $\lim_{t\to\infty} T(t) = \infty$. Solution of $X'' + k^2\, X = 0$ gives

$$X(x) = c_1 \cos(kx) + c_2 \sin(kx).$$

Left boundary condition $u(0,t) = 0$ implies $X(0) = 0 \Rightarrow c_1 = 0$. Right boundary condition $X(\pi) = 0$ leads to

$$c_2 \sin(k\pi) = 0,$$

satisfied for eigenvalues $k \in \mathrm{N}_+$. The value $k = 0$ is excluded since it implies $T(t) = c_3$, $X(x) = c_4 + c_5\, x$, and homogeneous boundary conditions on $X(x)$ would imply $c_4 = c_5 = 0 \Rightarrow X(x) = 0$.

The solution is therefore the series

$$u(x,t) = \sum_{k=1}^{\infty} A_k\, e^{-k^2 t} \sin(kx).$$

The initial condition

$$u(x,0) = \sin(x) = \sum_{k=1}^{\infty} A_k \sin(kx),$$

implies $A_1 = 1$, $A_k = 0$ for $k > 1$, hence the problem solution is

$$u(x,t) = e^{-t} \sin(x).$$