MATH529L02: Heat equation

One-dimensional heat equation

Dirichlet boundary conditions

Solve $u_t = k\, u_{xx}$ with $u(0, t) = 0$, $u(L, t) = 0$, $u(x, 0) = 100$ for $0 < x < L$, $k = 1$ (1)

In[1]:=
```
nT = 40; (* Number of terms in sine series *)
```

In[2]:=

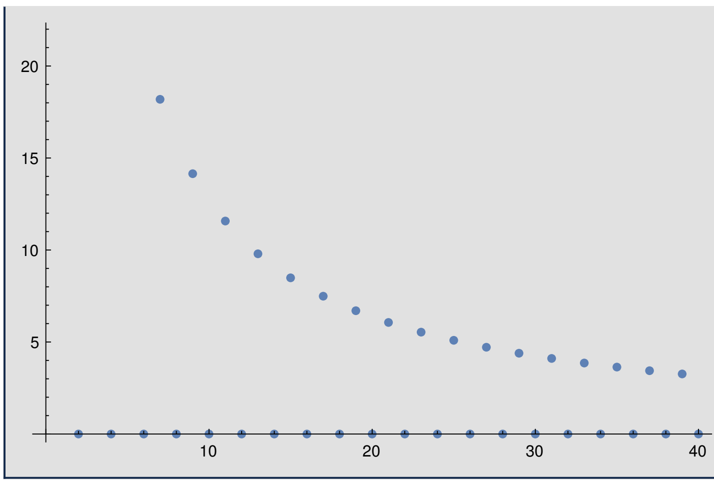
$$\alpha = \text{Table}\left[n \frac{\pi}{L}, \{n, 1, nT\}\right];$$

In[3]:=

$$b = \text{Table}\left[\frac{2}{L} \text{Integrate}[100 \text{Sin}[\alpha[[n]]\, x], \{x, 0, L\}], \{n, 1, nT\}\right];$$

In[4]:=
```
ListPlot[b]
```

Out[4]=



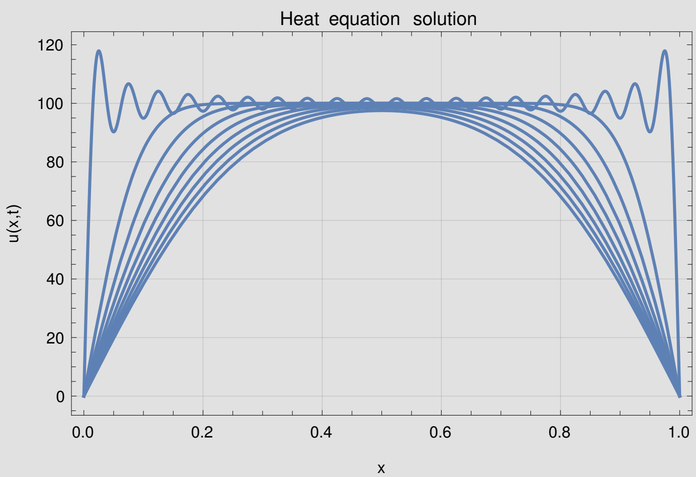
In[5]:=

$$u[x\_, t\_, L\_] = \text{Sum}\left[b[[n]] \text{Sin}[\alpha[[n]]\, x] \text{Exp}\left[- \alpha[[n]]^2\, t\right], \{n, 1, nT\}\right];$$

In[ ]:=

```
Plot[Table[u[x, t, 1], {t, 0, 0.02, 0.0025}], {x, 0, 1},
 PlotRange → All, Frame → True, PlotLabel → "Heat equation solution",
 FrameLabel → {"x", "u(x,t)"}, GridLines → Automatic]
```

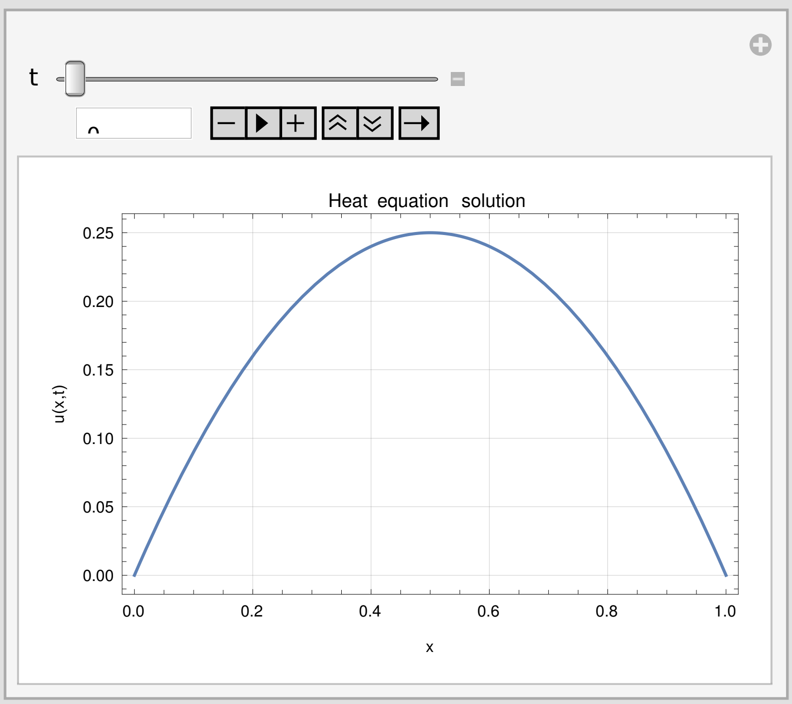
Out[ ]=



In[ ]:=

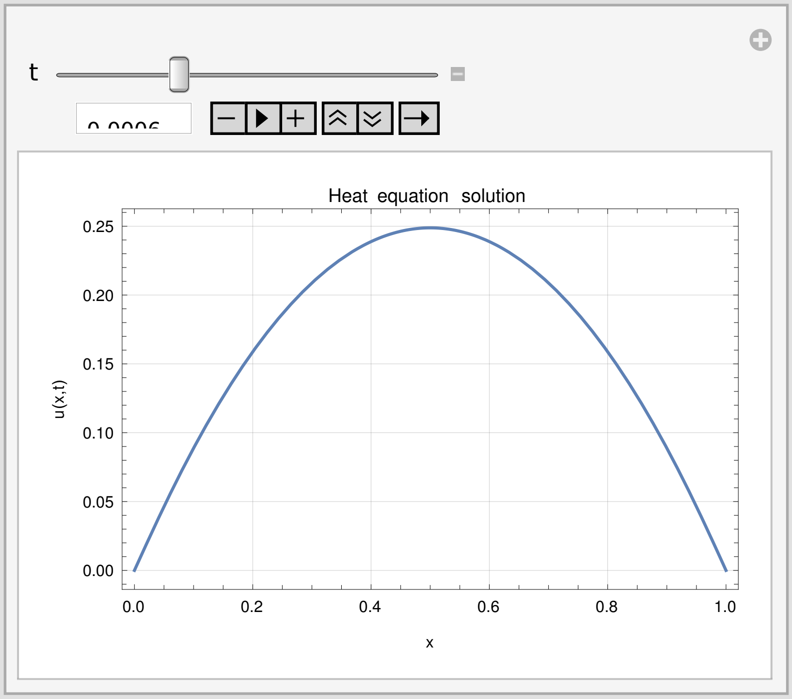
```
Manipulate[Plot[u[x, t, 1], {x, 0, 1}, PlotRange → All,
  Frame → True, PlotLabel → "Heat equation solution",
  FrameLabel → {"x", "u(x,t)"}, GridLines → Automatic], {t, 0, 0.02, 0.0025}]
```

Out[ ]=



In[ ]:=
```
Manipulate[Plot[u[x, t, 1], {x, 0, 1}, PlotRange → All,
  Frame → True, PlotLabel → "Heat equation solution",
  FrameLabel → {"x", "u(x,t)"}, GridLines → Automatic], {t, 0, 0.002, 0.0001}]
```

Out[ ]=



Neumann boundary conditions

Solve $u_t = k\, u_{xx}$ Exercise 2 : $u(0, t) = 0,\ u(L, t) = 0,\ u(x, 0) = x (L - x)$ for $0 < x < L,\ k = 1$ (2)

In[ ]:=
```
nT = 40; (* Number of terms in sine series *)
```

In[7]:=
```
α = Table[n π/L, {n, 1, nT}];
```

In[13]:=
```
bF[a_] = Integrate[x (L - x) Sin[a x], {x, 0, L}]
```

Out[13]=

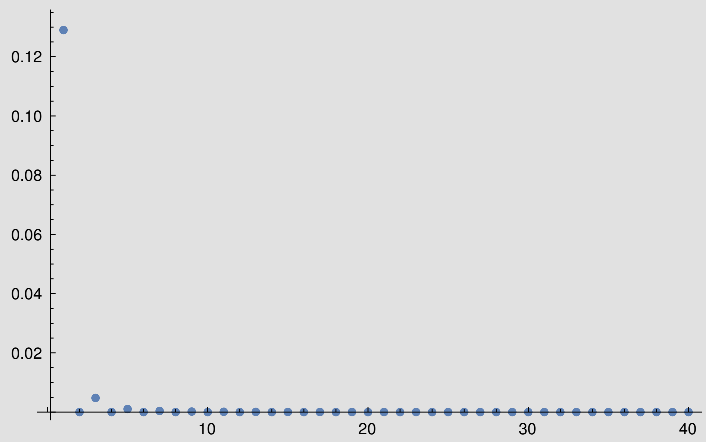
$$\frac{2 - 2\,\mathrm{Cos}[a\,L] - a\,L\,\mathrm{Sin}[a\,L]}{a^3}$$

In[15]:=
```
b = Table[bF[α[[n]]], {n, 1, nT}];
```

In[16]:=

```
ListPlot[b /. L → 1, PlotRange → All]
```

Out[16]=



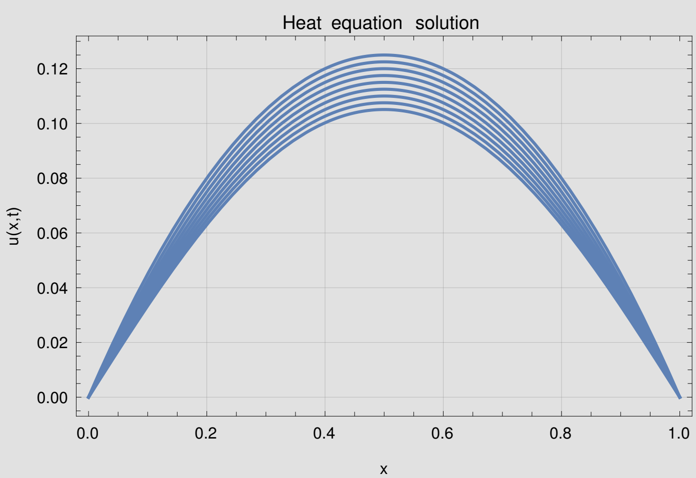
In[17]:=

```
u[x_, t_, L_] = Sum[ b[[n]] Sin[ α[[n]] x ] Exp[ - α[[n]]^2 t], {n, 1, nT}];
```

In[18]:=

```
Plot[Table[u[x, t, 1], {t, 0, 0.02, 0.0025}], {x, 0, 1},
 PlotRange → All, Frame → True, PlotLabel → "Heat equation solution",
 FrameLabel → {"x", "u(x,t)"}, GridLines → Automatic]
```

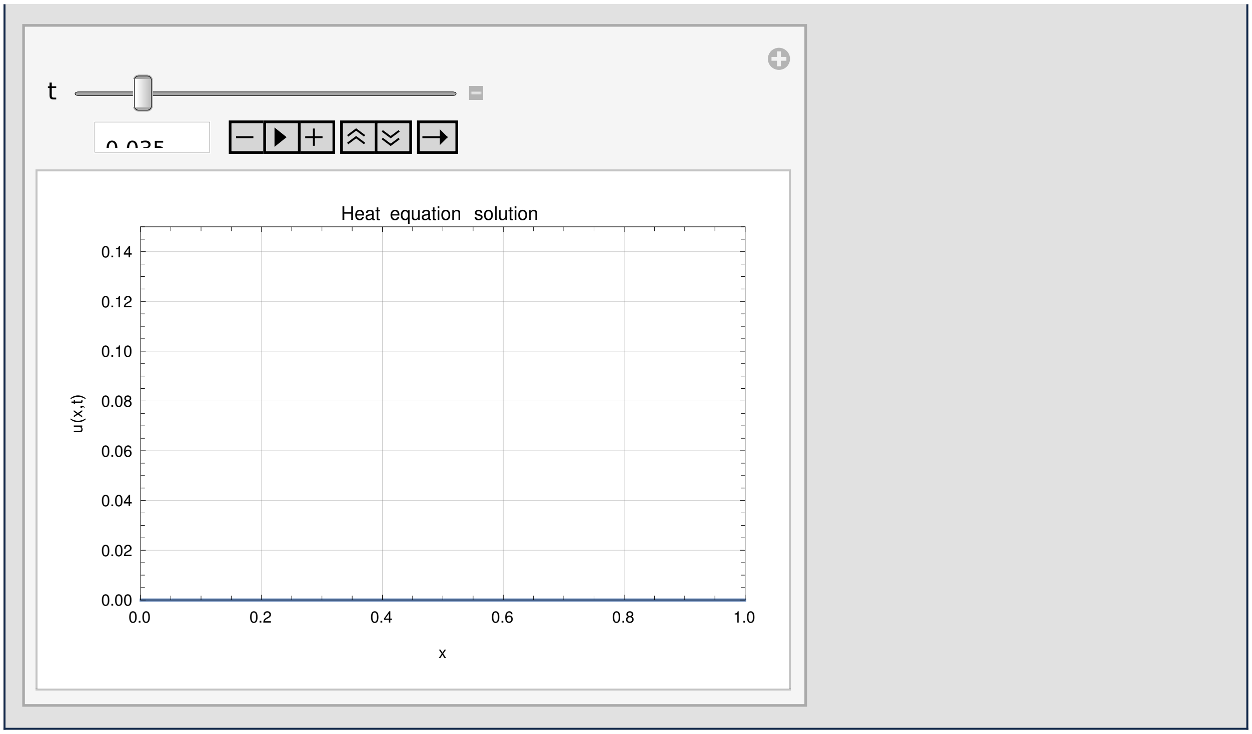
Out[18]=



In[23]:=

```
Manipulate[Plot[u[x, t, 1], {x, 0, 1}, Frame → True,
  PlotLabel → "Heat equation solution", FrameLabel → {"x", "u(x,t)"},
  GridLines → Automatic, PlotRange → {{0, 1}, {0, 0.15}}], {t, 0, 0.25, 0.005}]
```

Out[23]=



General : Exp[−710.612] is too small to represent as a normalized machine number ; precision may be lost .

General : Exp[−967.221] is too small to represent as a normalized machine number ; precision may be lost .

General : Exp[−1263.31] is too small to represent as a normalized machine number ; precision may be lost .

General : Further output of General ::munfl will be suppressed during this calculation .

General : Exp[−710.612] is too small to represent as a normalized machine number ; precision may be lost .

General : Exp[−967.221] is too small to represent as a normalized machine number ; precision may be lost .

General : Exp[−1263.31] is too small to represent as a normalized machine number ; precision may be lost .

General : Further output of General ::munfl will be suppressed during this calculation .

General : Exp[−809.308] is too small to represent as a normalized machine number ; precision may be lost .

General : Exp[−1204.09] is too small to represent as a normalized machine number ; precision may be lost .

General : Exp[−1677.83] is too small to represent as a normalized machine number ; precision may be lost .

General : Further output of General ::munfl will be suppressed during this calculation .

Two-dimensional heat equation

$u_t = k\,(u_{xx} + u_{yy})$, $u(x, y, t = 0) = 100$ on the $[0, 1] \times [0, 1]$ unit square, with zero temperature on edges (3)

Dirichlet boundary conditions

In[ ]:=
```
nT = 40; (* Number of terms in sine series *)
```

In[26]:=
```
α = Table[m π, {m, 1, nT}]; β = Table[ n π, {n, 1, nT}];
```

In[44]:=
```
bF[u_, v_] = 4 Integrate[100 Sin[u x] Sin[v y], {x, 0, 1}, {y, 0, 1}]
```

Out[44]=

$$\frac{1600\ \mathrm{Sin}\left[\frac{u}{2}\right]^2\ \mathrm{Sin}\left[\frac{v}{2}\right]^2}{u\ v}$$

In[45]:=
```
bF[u_] = Limit[bF[u, v], {v → u}]
```

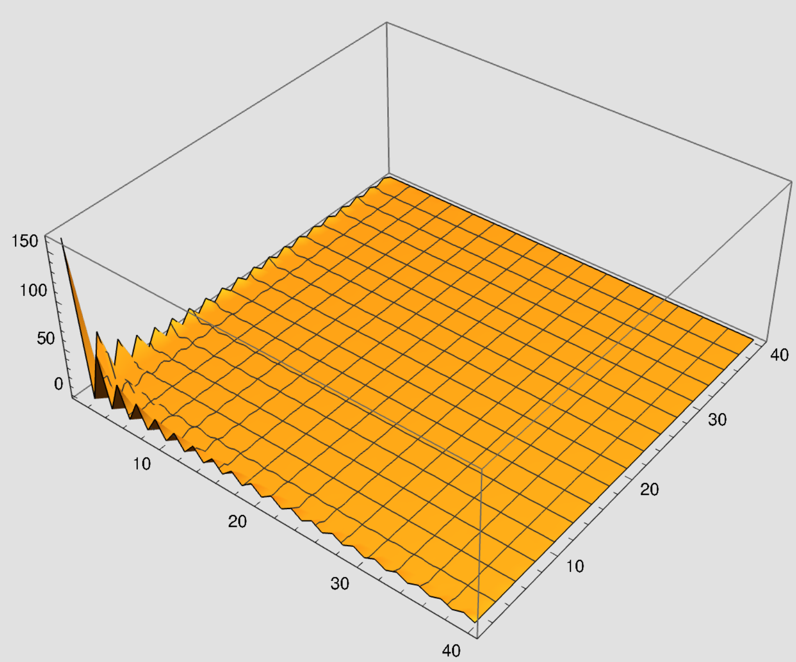
Out[45]=

$$\frac{1600\ \mathrm{Sin}\left[\frac{u}{2}\right]^4}{u^2}$$

In[47]:=
```
b = Table[If[m == n, bF[α[[m]]], bF[α[[m]], β[[n]]]], {m, 1, nT}, {n, 1, nT}];
```

In[52]:=
```
ListPlot3D[b, PlotRange → All]
```

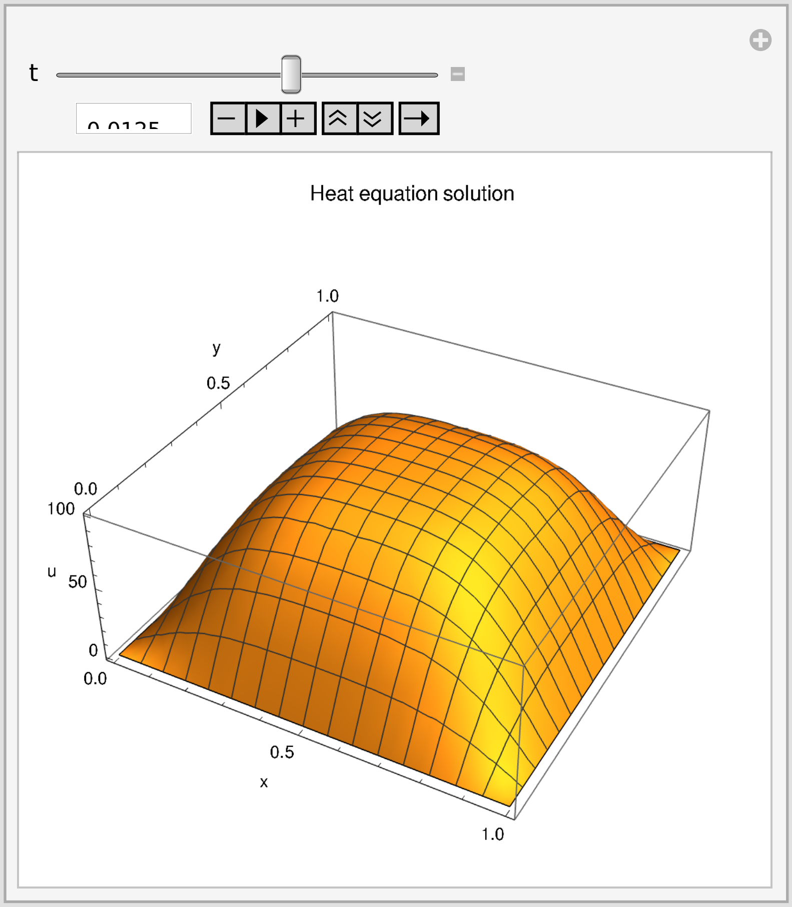
Out[52]=



In[54]:=
```
u[x_, y_, t_] =
  Sum[ b[[m, n]] Sin[ α[[m]] x ] Sin[ β[[n]]  y ] Exp[ -( α[[m]]^2 +  β[[n]]^2) t], {m, 1, nT}, {n, 1, nT}];
```

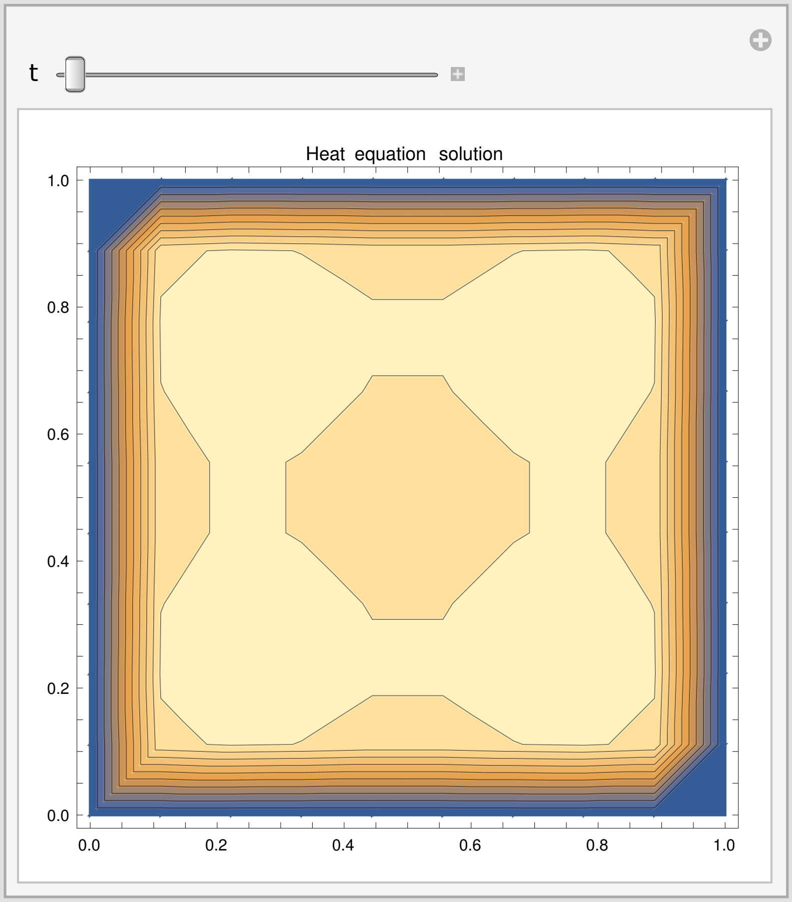
In[41]:=
```
Manipulate[Plot3D[u[x, y, t], {x, 0, 1}, {y, 0, 1},
  PlotRange → All, Axes → True, PlotLabel → "Heat equation solution",
  AxesLabel → {"x", "y", "u"}], {t, 0, 0.02, 0.0025}]
```

Out[41]=



In[56]:=
```
Manipulate[ContourPlot[u[x, y, t], {x, 0, 1}, {y, 0, 1}, PlotRange → All, Axes → True,
  PlotLabel → "Heat equation solution", AxesLabel → {"x", "y", "u"},
  Contours → Table[temp, {temp, 0, 100, 10}]], {t, 0, 0.02, 0.0025}]
```

Out[56]=



Questions to investigate

Dirichlet BVP with no edge discontinuity

$u_t = k(u_{xx} + u_{yy})$, $u(x, y, t = 0) = xy(1 - x)(1 - y)$ on the $[0, 1] \times [0, 1]$ unit square, with zero temperature on edges (4)

Neumann BVP with edge discontinuity

$u_t = k(u_{xx} + u_{yy})$, $u(x, y, t = 0) = 100$ on the $[0, 1] \times [0, 1]$ unit square, with zero temperature on $x = 0$, $x = 1$ vertical edges, $u_y(x, y = 0, t) = 0$, $u_y(x, y = 1, t) = 0$ (5)

Neumann BVP with no edge discontinuity

$$u_t = k\,(u_{xx} + u_{yy}),\ \ u(x, y, t = 0) = xy\,(1 - x)(1 - y) \text{ on the } [0, 1] \times [0, 1] \text{ unit square },\ ,$$
$$u_x(x = 0, y, t) = 0,\ u_x(x = 1, y, t) = 0,\ u_y(x, y = 0, t) = 0,\ u_y(x, y = 1, t) = 0 \qquad (6)$$