MATH529 Lab04:
Heat transfer in cylinder head fins

1 Theory

Reciprocating, internal combustion engines dissipate heat into the environment through specific systems to eliminate thermal stress in the engine.

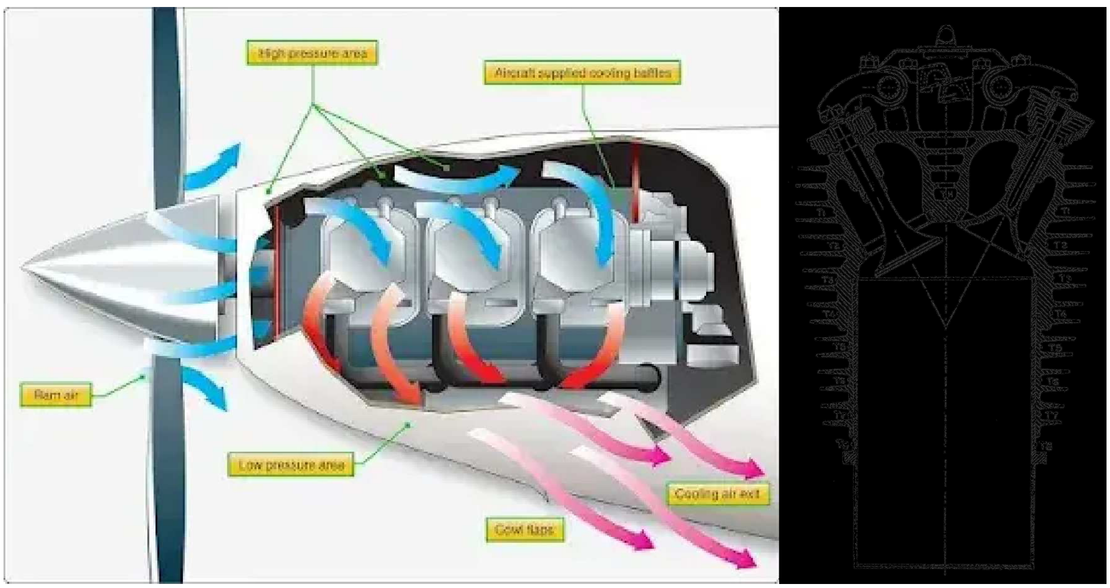

Figure 1. Aircraft engine schematic showing cooling flow [?]

2 Methods

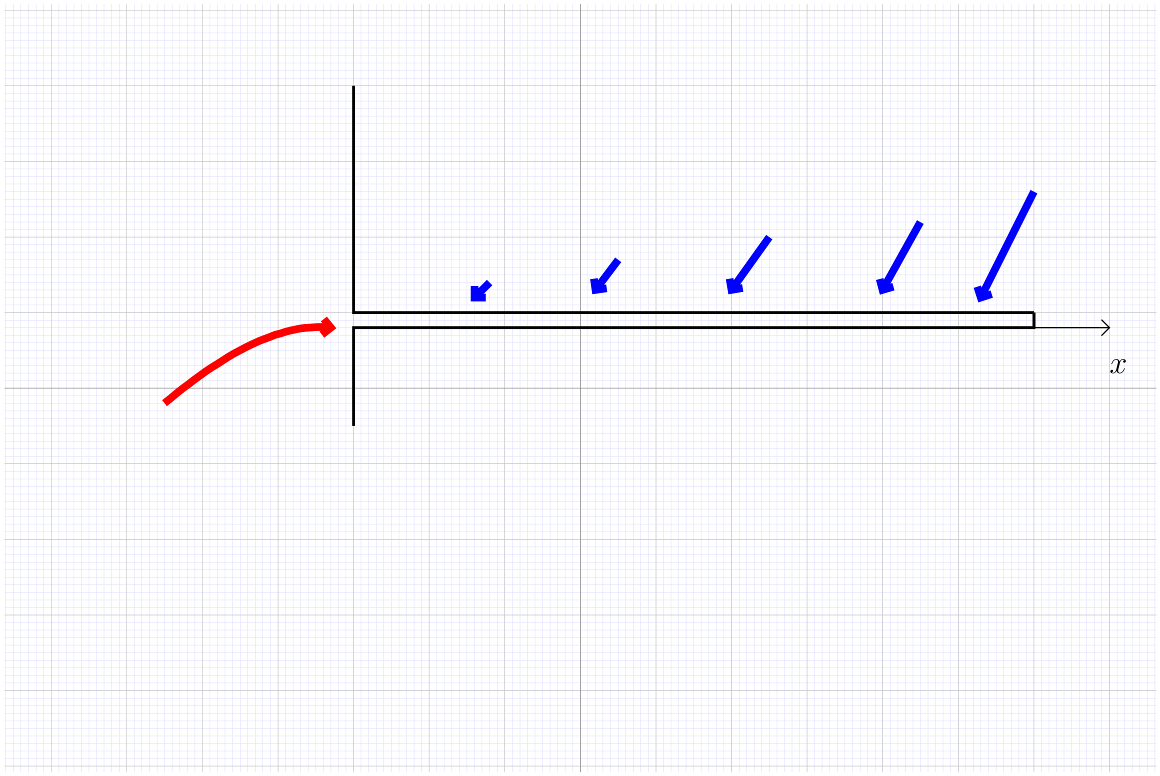

Figure 2. Model system

2.1 Dirichlet boundary conditions

Mathematical problem

$$\begin{array}{ll} u_t = k u_{xx} + q(x,t) & 0 < x < L \\ u(0,t) = u_0(t) = U_0 + U\cos(\Omega t), & u(L,t) = 0 \text{ (ambient temperature)} \\ u(x,0) = 0, & q(x,t) = \dfrac{x}{L} V\,[1 + W\cos(\omega t)] \text{ (convective cooling)} \end{array} \tag{1}$$

The homogeneous problem

$$\begin{array}{ll} w_t = k w_{xx} & 0 < x < L \\ w(0,t) = 0, & w(L,t) = 0 \text{ (ambient temperature)} \end{array}$$

has eigenfunctions

$$y_n(x) = \sin(n\pi x)$$

Expand $u(x,t)$ and $q(x,t)$ in terms of these eigenfunctions with time-dependent coefficients

$$u(t,x) = \sum_{n=1}^{\infty} u_n(t)\sin(n\pi x),\, q(t,x) = \sum_{n=1}^{\infty} q_n(t)\sin(n\pi x). \tag{2}$$

Replacing (2) into (1) obtain

$$\left(\sum_{n=1}^{\infty} u_n(t)\sin(n\pi x)\right)_t = k\left(\sum_{n=1}^{\infty} u_n(t)\sin(n\pi x)\right)_{xx} + \sum_{n=1}^{\infty} q_n(t)\sin(n\pi x)$$

$$\sum_{n=1}^{\infty} \dot{u}_n(t)\sin(n\pi x) = -k\pi^2\sum_{n=1}^{\infty} n^2 u_n(t)\sin(n\pi x) + \sum_{n=1}^{\infty} q_n(t)\sin(n\pi x)$$

Since $\{\sin(\pi x), \sin(2\pi x), \dots\}$ is linearly independent

$$\sum_{n=1}^{\infty} [\dot{u}_n(t) + k\pi^2 n^2 u_n(t) - q_n(t)]\sin(n\pi x) = 0 \Rightarrow \dot{u}_n + k\pi^2 n^2 u_n - q_n = 0,$$

an inhomogeneous system of ODEs

Obtain $q_n(t)$ by Fourier coefficients of

$$\sum_{n=1}^{\infty} q_n(t)\sin(n\pi x) = q(x,t) = \frac{x}{L} V\,[1 + W\cos(\omega t)] \Rightarrow$$

$$q_n(t) = \frac{2V}{L^2}[1 + W\cos(\omega t)]\int_0^L x\sin(n\pi x)\,\mathrm{d}x \Rightarrow A_n = \frac{\sin(\pi L n) - \pi L n\cos(\pi L n)}{\pi^2 n^2}.$$

$$q_n(x,t) = A_n\frac{2V}{L^2}[1 + W\cos(\omega t)]$$

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In[40]:= A[n_]=Integrate[x Sin[n Pi x],{x,0,L}]
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$$\frac{\sin(\pi L n) - \pi L n\cos(\pi L n)}{\pi^2 n^2}$$

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In[41]:=
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- The system to solve is

$$\dot{u}_n + k\pi^2 n^2 u_n = A_n\frac{2V}{L^2}[1 + W\cos(\omega t)] = B_n$$

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In[42]:= sol=DSolve[{D[un[t],t]+k Pi^2 n^2 un[t] == B[n] (1+ Cos[omega t]),
         un[0]==U0},un[t],t][[1,1]]
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$$\mathrm{un}(t)\to e^{-kn^2\pi^2 t}(k^5 L^2\pi^{12}\mathrm{U0}n^{12}-3e^{kn^2\pi^2 t}k^4 L\pi^9 V\cos(Ln\pi)n^9+8k^4 L\pi^9 V\cos(Ln\pi)n^9-4e^{kn^2\pi^2 t}k^4 L\pi^9 V\cos(Ln\pi)\cos(\omega t)n^9-e^{kn^2\pi^2 t}k^4 L\pi^9 V\cos(Ln\pi)\cos(2\omega t)n^9+5k^3L^2\omega^2\pi^8\mathrm{U0}n^8+3e^{kn^2\pi^2 t}k^4\pi^8 V\sin(Ln\pi)n^8-8k^4\pi^8 V\sin(Ln\pi)n^8+4e^{kn^2\pi^2 t}k^4\pi^8 V\cos(\omega t)\sin(Ln\pi)n^8+e^{kn^2\pi^2 t}k^4\pi^8 V\cos(2\omega t)\sin(Ln\pi)n^8-4e^{kn^2\pi^2 t}k^3 L\omega\pi^7 V\cos(Ln\pi)\sin(\omega t)n^7-2e^{kn^2\pi^2 t}k^3 L\omega\pi^7 V\cos(Ln\pi)\sin(2\omega t)n^7+4e^{kn^2\pi^2 t}k^3\omega\pi^6 V\sin(Ln\pi)\sin(\omega t)n^6+2e^{kn^2\pi^2 t}k^3\omega\pi^6 V\sin(Ln\pi)\sin(2\omega t)n^6-15e^{kn^2\pi^2 t}k^2 L\omega^2\pi^5 V\cos(Ln\pi)n^5+32k^2 L\omega^2\pi^5 V\cos(Ln\pi)n^5-16e^{kn^2\pi^2 t}k^2 L\omega^2\pi^5 V\cos(Ln\pi)\cos(\omega t)n^5-e^{kn^2\pi^2 t}k^2 L\omega^2\pi^5 V\cos(Ln\pi)\cos(2\omega t)n^5+4kL^2\omega^4\pi^4\mathrm{U0}n^4+15e^{kn^2\pi^2 t}k^2\omega^2\pi^4 V\sin(Ln\pi)n^4-32k^2\omega^2\pi^4 V\sin(Ln\pi)n^4+16e^{kn^2\pi^2 t}k^2\omega^2\pi^4 V\cos(\omega t)\sin(Ln\pi)n^4+e^{kn^2\pi^2 t}k^2\omega^2\pi^4 V\cos(2\omega t)\sin(Ln\pi)n^4-16e^{kn^2\pi^2 t}kL\omega^3\pi^3 V\cos(Ln\pi)\sin(\omega t)n^3-2e^{kn^2\pi^2 t}kL\omega^3\pi^3 V\cos(Ln\pi)\sin(2\omega t)n^3+16e^{kn^2\pi^2 t}k\omega^3\pi^2 V\sin(Ln\pi)\sin(\omega t)n^2+2e^{kn^2\pi^2 t}k\omega^3\pi^2 V\sin(Ln\pi)\sin(2\omega t)n^2-12e^{kn^2\pi^2 t}L\omega^4\pi V\cos(Ln\pi)n+12L\omega^4\pi V\cos(Ln\pi)n+12e^{kn^2\pi^2 t}\omega^4 V\sin(Ln\pi)-12\omega^4 V\sin(Ln\pi))/(kL^2n^4\pi^4(k^2\pi^4n^4+\omega^2)(k^2\pi^4n^4+4\omega^2))$$

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In[43]:= u[x_,t_,nT_] := Sum[ un[t] Sin[n Pi x] /. sol, {n,1,nT}]
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In[44]:= u[x,t,1]
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$$e^{\pi^2(-k)t}\sin(\pi x)(\pi^{12}k^5L^2\mathrm{U0}-4\pi^9k^4LVe^{\pi^2kt}\cos(\pi L)\cos(\omega t)-\pi^9k^4LVe^{\pi^2kt}\cos(\pi L)\cos(2\omega t)+4\pi^8k^4Ve^{\pi^2kt}\sin(\pi L)\cos(\omega t)+\pi^8k^4Ve^{\pi^2kt}\sin(\pi L)\cos(2\omega t)+3\pi^8k^4Ve^{\pi^2kt}\sin(\pi L)-3\pi^9k^4LVe^{\pi^2kt}\cos(\pi L)-8\pi^8k^4V\sin(\pi L)+8\pi^9k^4LV\cos(\pi L)+5\pi^8k^3L^2\omega^2\mathrm{U0}+4\pi^6k^3\omega Ve^{\pi^2kt}\sin(\pi L)\sin(\omega t)+2\pi^6k^3\omega Ve^{\pi^2kt}\sin(\pi L)\sin(2\omega t)-4\pi^7k^3L\omega Ve^{\pi^2kt}\cos(\pi L)\sin(\omega t)-2\pi^7k^3L\omega Ve^{\pi^2kt}\cos(\pi L)\sin(2\omega t)+15\pi^4k^2\omega^2Ve^{\pi^2kt}\sin(\pi L)-15\pi^5k^2L\omega^2Ve^{\pi^2kt}\cos(\pi L)-16\pi^5k^2L\omega^2Ve^{\pi^2kt}\cos(\pi L)\cos(\omega t)-\pi^5k^2L\omega^2Ve^{\pi^2kt}\cos(\pi L)\cos(2\omega t)+16\pi^4k^2\omega^2Ve^{\pi^2kt}\sin(\pi L)\cos(\omega t)+\pi^4k^2\omega^2Ve^{\pi^2kt}\sin(\pi L)\cos(2\omega t)-32\pi^4k^2\omega^2V\sin(\pi L)+32\pi^5k^2L\omega^2V\cos(\pi L)+4\pi^4kL^2\omega^4\mathrm{U0}+12\omega^4Ve^{\pi^2kt}\sin(\pi L)-12\pi L\omega^4Ve^{\pi^2kt}\cos(\pi L)+16\pi^2k\omega^3Ve^{\pi^2kt}\sin(\pi L)\sin(\omega t)+2\pi^2k\omega^3Ve^{\pi^2kt}\sin(\pi L)\sin(2\omega t)-16\pi^3kL\omega^3Ve^{\pi^2kt}\cos(\pi L)\sin(\omega t)-2\pi^3kL\omega^3Ve^{\pi^2kt}\cos(\pi L)\sin(2\omega t)-12\omega^4V\sin(\pi L)+12\pi L\omega^4V\cos(\pi L))/(\pi^4kL^2(\pi^4k^2+\omega^2)(\pi^4k^2+4\omega^2))$$

```
In[46]:= u[x,t,1] /. {k->1,L->1,omega->1,OMEGA->5,V->1,U0->2,U->1}
```

$$e^{-\pi^2 t}\sin(\pi x)(3\pi^9e^{\pi^2t}+15\pi^5e^{\pi^2t}+12\pi e^{\pi^2t}+4\pi^7e^{\pi^2t}\sin(t)+16\pi^3e^{\pi^2t}\sin(t)+2\pi^7e^{\pi^2t}\sin(2t)+2\pi^3e^{\pi^2t}\sin(2t)+4\pi^9e^{\pi^2t}\cos(t)+16\pi^5e^{\pi^2t}\cos(t)+\pi^9e^{\pi^2t}\cos(2t)+$$

$\pi^5 e^{\pi^2 t} \cos(2t) + 2\pi^{12} - 8\pi^9 + 10\pi^8 - 32\pi^5 + 8\pi^4 - 12\pi)/(\pi^4(1+\pi^4)(4+\pi^4))$

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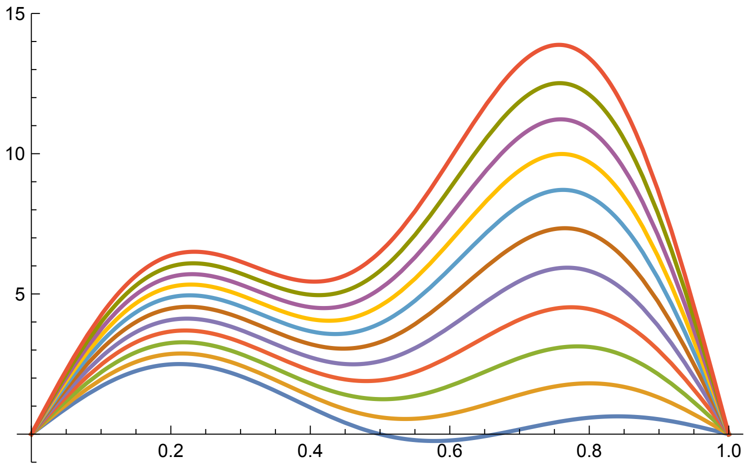
In[54]:= Plot[Evaluate[Table[u[x,t,3] /. {k->0.001,L->1,omega->180,OMEGA->30,V->10,U0->1,U->1},{t,0,1,0.1}]],{x,0,1},PlotRange->{All,{-1,15}}]
```




```
In[22]:= Evaluate[Table[u[x,t,5] /. {k->1,L->1,omega->1,Cn->1/n^2,c[1]->0},{t,0,0.01,0.05}]]
```

$$\left\{\left(\frac{0.201613}{n^2} + 1.\, c_1\right)\sin(\pi x) + \left(\frac{0.0506443}{n^2} + 1.\, c_1\right)\sin(2\pi x) + \left(\frac{0.0225144}{n^2} + 1.\, c_1\right)\sin(3\pi x) + \left(\frac{0.0126649}{n^2} + 1.\, c_1\right)\sin(4\pi x) + \left(\frac{0.00810563}{n^2} + 1.\, c_1\right)\sin(5\pi x)\right\}$$

```
In[8]:= u[x,t,3] /. {k->1,L->1,omega->1,Cn->1/n^2,c[1]->0}
```

$$\frac{e^{-\pi^2 t}\sin(\pi x)\,(e^{\pi^2 t} + \pi^4 e^{\pi^2 t} + \pi^2 e^{\pi^2 t}\sin(t) + \pi^4 e^{\pi^2 t}\cos(t) - 2\pi^4 - 1)}{\pi^2(1+\pi^4)\,n^2} +$$

$e^{-4\pi^2 t}\sin(2\pi x)\,(e^{4\pi^2 t} + 16\pi^4 e^{4\pi^2 t} + 4\pi^2 e^{4\pi^2 t}\sin(t) + 16\pi^4 e^{4\pi^2 t}\cos(t) - 32\pi^4 - 1)/(4\pi^2(1+16\pi^4)\,n^2) + e^{-9\pi^2 t}\sin(3\pi x)\,(e^{9\pi^2 t} + 81\pi^4 e^{9\pi^2 t} + 9\pi^2 e^{9\pi^2 t}\sin(t) + 81\pi^4 e^{9\pi^2 t}\cos(t) - 162\pi^4 - 1)/(9\pi^2(1+81\pi^4)\,n^2)$

```
In[9]:=
```

Introduce

$$\psi(x,t) = u_0(t)\, q(x,t),$$

that satisfies

$$\psi_{xx} = 0$$

and $v = u - \psi$ leads to the problem

$$\begin{array}{ll} v_t = k v_{xx} + q(x,t) - \psi_t & 0 < x < L \\ v(0,t) = u_0(t) = U_0 + U\cos(\Omega t), & u(L,t) = 0 \text{ (ambient temperature)} \\ u(x,0) = 0, & q(x,t) = \frac{x}{L} V\,[1 + W\cos(\omega t)] \text{ (convective cooling)} \end{array}$$

2.2 Robin boundary condition

$$\begin{array}{ll} u_t = k u_{xx} + q(x,t) & 0 < x < L \\ \alpha[u(0,t) - u_0] = k u_x(0,t), & u(L,t) = 0 \text{ (ambient temperature)} \\ u(x,0) = 0, & q(x,t) = \frac{x}{L} V\,[1 + W\cos(\omega t)] \text{ (convective cooling)} \end{array} \quad (3)$$

3 Results

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Figure 3.

```
In[58]:= fig=Plot[Cos[x],{x,0,2Pi}];

In[61]:= SetDirectory["/Users/mitran/courses/MATH529L"]

         /Users/mitran/courses/MATH529L

In[62]:= Export["fig.png",fig]

         fig.png

In[64]:= fig=Plot[Cos[x],{x,0,2Pi},GridLines->Automatic];
         Export["fig.png",fig];

In[65]:=
```

4 Conclusion

Bibliography