- SIR space-time model
- Graphs
- Calculus on graphs
- Graph Laplacian, diffusion

- Recall coupled PDE SIR model

$$
\begin{aligned}
S'(t) &= -\beta I(t)\, S(t) \\
I'(t) &= \beta I(t)\, S(t) - \gamma I(t) \\
R' &= \gamma I \\
S, I, R &: \mathbb{R} \to \mathbb{R}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial S}{\partial t} &= -\beta S\,(I + \alpha \nabla^2 I) + \delta \nabla^2 S \\
\frac{\partial I}{\partial t} &= \beta S\,(I + \alpha \nabla^2 I) - \gamma I + \epsilon \nabla^2 I \\
\frac{\partial R}{\partial t} &= \varphi \nabla^2 R + \gamma I \\
S, I, R &: \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}
\end{aligned}
$$

- Form four-parameter reduced SI model

$$
\begin{aligned}
\frac{\partial S}{\partial t} &= -\beta S\,(I + \alpha \nabla^2 I) + \delta \nabla^2 S \\
\frac{\partial I}{\partial t} &= \beta S\,(I + \alpha \nabla^2 I) - \gamma I \\
S, I &: \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}
\end{aligned}
$$

- Graph: a set of vertices and edges, $G = (V, E)$
- Graph connectivity matrix, $\boldsymbol{C} \in \mathbb{R}^{m \times m}$, $m = |V|$

$$\boldsymbol{C} = [C_{ij}]_{1 \leqslant i,j \leqslant m}, C_{ij} \neq 0 \text{ if } V_i \text{ connected to } V_j$$

  - $C_{ij} \in \{0, 1\}$, simple connectivity data, neighbor of relation, $x_i \sim x_j$
  - $C_{ij} \in \mathbb{R}$, additional data, e.g., distance, flux, ... In general some "weight" associated with edge, hence notation $\boldsymbol{W} = [w_{ij}]_{1 \leqslant i,j \leqslant m}$
- Graphs can represent both:
  - discretizations of continua, e.g., Cartesian grid within $\mathbb{R}^2$
  - disperse data, e.g., positions of cities, counties in the US
- Similar to calculus in $\mathbb{R}, \mathbb{R}^2, \ldots$, a calculus can be define for functions defined on a graph

- Differentiation, $f\colon \mathbb{R} \to \mathbb{R}$, $g\colon G \to \mathbb{R}$

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}, \quad \frac{\partial g(x_i)}{\partial x_j} = \sqrt{w_{ij}}\,[g(x_j) - g(x_i)]$$

- Gradient, $f\colon \mathbb{R}^2 \to \mathbb{R}$, $g\colon G \to \mathbb{R}$

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right), \quad \nabla_w g(x_i) = \left(\frac{\partial g}{\partial x_j}\right)_{j \sim i}$$

- Divergence, $\boldsymbol{v}\colon \mathbb{R}^2 \to \mathbb{R}^2$, $F\colon G \to \mathbb{R}$, $F_{ij} = F(x_j, x_i)$

$$\nabla \cdot \boldsymbol{v} = \lim_{|\Omega| \to 0} \frac{1}{|\Omega|} \oint_{S=\partial\Omega} \boldsymbol{v} \cdot \boldsymbol{n}\, \mathrm{d}S, \quad \nabla_w^* F(x_i) = \sum_{x_j \sim x_i} \sqrt{w_{ij}}\,[F_{ij} - F_{ji}]$$

- Laplacian in $\mathbb{R}^2$, $\nabla^2 f = \nabla \cdot (\nabla f)$
- Laplacian in $G = (V, E)$

$$\Delta_w f(x_i) = \nabla_w^2 f(x_i) = \sum_{x_j \sim x_i} w_{ij} \left(f(x_j) - f(x_i) \right)$$

- Diffusion in $\mathbb{R}^2$

$$\frac{\partial f}{\partial t} = \alpha \nabla^2 f$$

- Graph diffusion equation

$$\frac{\mathrm{d} f(x_i)}{\mathrm{d} t} = \alpha \nabla_w^2 f(x_i) = \alpha \sum_{x_j \sim x_i} w_{ij} \left(f(x_j) - f(x_i) \right)$$