MATH547 Final Examination

Fall 2015 Semester, December 11, 2015

Instructions. *Answer the following questions. Provide a motivation of your approach and the reasoning underlying successive steps in your solution. Write neatly and avoid erasures. Use scratch paper to sketch out your answer for yourself, and then transcribe your solution to the examination you turn in for grading. Illegible answers are not awarded any credit. Presentation of calculations without mention of the motivation and reasoning are not awarded any credit. The last, seventh question is optional and offered to enable raising your score on the midterm examination. Each complete, correct solution to an examination question is awarded 4 course grade points. Your primary goal should be to demonstrate understanding of course topics and skill in precise mathematical formulation and solution procedures.*

1. Consider the orthogonal matrix $\boldsymbol{Q} \in \mathbb{R}^{m \times m}$, $\boldsymbol{Q}^T \boldsymbol{Q} = \boldsymbol{I}$. Prove that $\det \boldsymbol{Q} = \pm 1$.

2. Consider the matrix $\boldsymbol{V} \in \mathbb{R}^{3 \times 3}$ and cubic polynomial $p(t)$ defined by

$$\boldsymbol{V} = \begin{pmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{pmatrix}, p(t) = \det \begin{pmatrix} 1 & t & t^2 & t^3 \\ 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \end{pmatrix}.$$

   (a) Compute $\det \boldsymbol{V}$.

   (b) What are the roots of $p(t)$?

3. Compute $\boldsymbol{A}^{16}$ for

$$\boldsymbol{A} = \begin{pmatrix} 4 & -3 \\ 2 & -1 \end{pmatrix}.$$

4. Let λ be an eigenvalue of the nonsingular matrix $\boldsymbol{A} \in \mathbb{R}^{m \times m}$. Prove that λ^{-1} is an eigenvalue of $\boldsymbol{A}^{-1}$.

5. Compute the singular value decomposition of

$$\boldsymbol{A} = \begin{pmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{pmatrix}.$$

6. Consider the ordinary differential equation system $\boldsymbol{x}'(t) = \boldsymbol{A}\boldsymbol{x}(t)$ with $\boldsymbol{x}(t)\colon \mathbb{R} \to \mathbb{R}^2$, $\boldsymbol{A} \in \mathbb{R}^{2 \times 2}$ with eigenvalues $\lambda_1 = -3$, $\lambda_2 = -1$, and corresponding eigenvectors

$$\boldsymbol{v}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \boldsymbol{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

   Compute solution at $t = 1$, $\boldsymbol{x}(1)$ given the initial condition at $t = 0$,

$$\boldsymbol{x}(0) = \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

7. Consider $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ with the property that $\boldsymbol{A}^T \boldsymbol{A}$ is nonsingular. Prove that the columns of $\boldsymbol{A}$ are linearly independent.