MATH547 Final Examination

Spring 2017 Semester, May 4, 2017

Instructions. *Answer the following questions. Provide a motivation of your approach and the reasoning underlying successive steps in your solution. Write neatly and avoid erasures. Use scratch paper to sketch out your answer for yourself, and then transcribe your solution to the examination you turn in for grading. Illegible answers are not awarded any credit. Presentation of calculations without mention of the motivation and reasoning are not awarded any credit. Each complete, correct solution to an examination question is awarded 4 course grade points. The last, seventh question is optional and offered to enable raising your score on the midterm examination. Your primary goal should be to demonstrate understanding of course topics and skill in precise mathematical formulation and solution procedures.*

1. Consider the matrix $\boldsymbol{Q} = (\ \boldsymbol{q}_1\ \ \boldsymbol{q}_2\ \ \dots\ \ \boldsymbol{q}_n\ ) \in \mathbb{R}^{m\times n}$ with orthonormal columns, $\boldsymbol{q}_i^T \boldsymbol{q}_j = \delta_{ij}$. Determine the eigenvalues and eigenvectors of the projection matrix $\boldsymbol{P} = \boldsymbol{Q}\boldsymbol{Q}^T$ onto $C(\boldsymbol{Q})$.

2. It is known that $\boldsymbol{A} \in \mathbb{R}^{3\times 3}$ has eigenvalues $\lambda_1 = 0$, $\lambda_2 = 1$, $\lambda_3 = 2$.

   a) Does the linear system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ ($\boldsymbol{x}, \boldsymbol{b} \in \mathbb{R}^3$) have a solution?

   b) Is the solution to the linear system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ unique?

   c) What are the eigenvalues of $(\boldsymbol{I} + \boldsymbol{A}^2)^{-1}$?

3. Determine $\boldsymbol{A}^{31}$ with

$$\boldsymbol{A} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

4. Prove that the eigenvectors of a real, symmetric matrix with distinct eigenvalues are orthogonal.

5. Consider $\boldsymbol{A} = (\ \boldsymbol{a}_1\ \ \boldsymbol{a}_2\ \ \dots\ \ \boldsymbol{a}_n\ ) \in \mathbb{R}^{m\times n}$ with orthogonal columns, $i \neq j \Rightarrow \boldsymbol{a}_i^T \boldsymbol{a}_j = \delta_{ij}$. Determine the singular value decomposition (SVD) of $\boldsymbol{A}$, i.e., $\boldsymbol{A} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^T$.

6. Compute the singular value decomposition of

$$\boldsymbol{A} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

7. Find bases for the four fundamental subspaces $C(\boldsymbol{A})$, $N(\boldsymbol{A}^T)$, $C(\boldsymbol{A}^T)$, $N(\boldsymbol{A})$ associated with the matrix

$$\boldsymbol{A} = \begin{pmatrix} 1 & -3 & 0 \\ 2 & -6 & 4 \\ -3 & 9 & 1 \end{pmatrix}.$$

Specify the rank and nullity of $\boldsymbol{A}$.