MATH547 Final Examination

Fall 2014 Semester, December 9, 2014

Instructions. *Answer the following questions. Provide a motivation of your approach and the reasoning underlying successive steps in your solution. Write neatly and avoid erasures. Use scratch paper to sketch out your answer for yourself, and then transcribe your solution to the examination you turn in for grading. Illegible answers are not awarded any credit. Presentation of calculations without mention of the motivation and reasoning are not awarded any credit. The last, seventh question is optional and offered to enable raising your score on the midterm examination. Each complete, correct solution to an examination question is awarded 4 course grade points. Your primary goal should be to demonstrate understanding of course topics and skill in precise mathematical formulation and solution procedures.*

1. Consider a block partitioning of a square matrix $M \in \mathbb{R}^{m \times m}$

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{1}$$

with A, B, C, D compatible submatrices. Compute

$$\begin{pmatrix} I & 0 \\ -CA^{-1} & I \end{pmatrix} M. \tag{2}$$

Solution. Carry out block multiplication

$$\begin{pmatrix} I & 0 \\ -CA^{-1} & I \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} IA + 0C & IB + 0D \\ -CA^{-1}A + C & -CA^{-1}B + D \end{pmatrix} = \begin{pmatrix} A & B \\ 0 & D - CA^{-1}B \end{pmatrix}$$

2. Again consider the block partitioning (1) of matrix $M \in \mathbb{R}^{m \times m}$. Use the result from (2) and the identity

$$\begin{vmatrix} A & B \\ 0 & D \end{vmatrix} = |A|\,|D| \tag{3}$$

to prove that if $AC = CA$ then

$$\det(M) = |M| = |AD - CB|.$$

Solution. Recall that determinant of matrix product is equal to product of matrix determinants and apply to result from Problem 1

$$\begin{vmatrix} I & 0 \\ -CA^{-1} & I \end{vmatrix} \begin{vmatrix} A & B \\ C & D \end{vmatrix} = \begin{vmatrix} A & B \\ 0 & D - CA^{-1}B \end{vmatrix}$$

Use (3) to obtain

$$\det(M) = |A|\,|D - CA^{-1}B|$$

Product of matrix determinants is determinant of matrix product rule gives

$$\det(M) = |A(D - CA^{-1}B)| = |AD - ACA^{-1}B|$$

If A, C commute ($AC = CA$) then

$$\det(M) = |AD - CB|.$$

3. Consider the matrix $V \in \mathbb{R}^{3\times 3}$

$$V = \begin{pmatrix} 1 & 1 & 3 \\ 1 & -2 & 0 \\ 1 & 1 & -3 \end{pmatrix}$$

with mutually orthogonal column vectors V_1, V_2, $V_3 \in \mathbb{R}^3$. What is the volume of the parallelepiped with edges V_1, V_2, V_3?

Solution. Since vectors are orthogonal the volume is simply the product of each vector norm

$$\mathcal{V} = \sqrt{3}\,\sqrt{6}\sqrt{18} = 18.$$

(Of course there are longer ways to do this also)

4. Find the eigenvalues and unit eigenvectors of

$$A = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}.$$

Solution. Columns 2,3 are repeated and linearly independent from column 1, so $\operatorname{rank}(A) = 2$ and $\lambda_3 = 0$ must be an eigenvalue and the row-echelon reduction

$$\begin{pmatrix} 2 & 2 & 2 \\ 2 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 2 & 2 & 2 \\ 0 & -2 & -2 \\ 0 & -2 & -2 \end{pmatrix}$$

leads to eigenvector $x_3 = \frac{1}{\sqrt{2}}(0, 1, -1)^T$. The characteristic polynomial of A is

$$\det(A - \lambda I) = \lambda^2(2 - \lambda) + 8\lambda = -\lambda(\lambda - 4)(\lambda + 2)$$

Row echelon reduction for $A - \lambda_2 I$, $\lambda_2 = -2$ gives

$$\begin{pmatrix} 4 & 2 & 2 \\ 2 & 2 & 0 \\ 2 & 0 & 2 \end{pmatrix} \sim \begin{pmatrix} 4 & 2 & 2 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

with resulting eigenvector $x_2 = \frac{1}{\sqrt{3}}(-1, 1, 1)^T$. Repeating for $\lambda_1 = 4$ gives

$$\begin{pmatrix} -2 & 2 & 2 \\ 2 & -4 & 0 \\ 2 & 0 & -4 \end{pmatrix} \sim \begin{pmatrix} -2 & 2 & 2 \\ 0 & -2 & 2 \\ 0 & 2 & -2 \end{pmatrix}$$

and $x_3 = \frac{1}{\sqrt{6}}(2, 1, 1)^T$.

5. Show that the eigenvalues of a skew-symmetric real matrix, $A \in \mathbb{R}^{m \times m}$, $A^T = -A$ are purely imaginary.

Solution. Consider the eigenvalue relation $Ax = \lambda x$, it's conjugate $A\,\bar{x} = \bar{\lambda}\,\bar{x}$, and the transpose $\bar{x}^T A^T = \bar{\lambda}\,\bar{x}^T = -\bar{x}^T A$. From these form scalar products

$$Ax = \lambda x \Rightarrow \bar{x}^T A x = \lambda \bar{x}^T x$$

$$\bar{x}^T A = -\bar{\lambda}\,\bar{x}^T \Rightarrow \bar{x}^T A x = -\bar{\lambda} \bar{x}^T x$$

Subtract to obtain

$$0 = (\lambda + \bar{\lambda}) \bar{x}^T x$$

But x is an eigenvector, hence $x \neq 0$ such that $\lambda + \bar{\lambda} = 0$. With $\lambda = u + iv$ deduce $u + iv + u - iv = 2u = 0$, hence λ has zero real part (purely imaginary)

6. Compute the singular value decomposition of

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

What is $\operatorname{rank}(A)$?

Solution. The matrix A is symmetric and real valued hence can be diagonalized $A = U \Lambda U^T$ with U an orthogonal matrix. The eigendecomposition is also a singular value decomposition $A = U \Sigma V^T$ with $\Sigma = \Lambda$ (assume eigenvalues are ordered in diminishing magnitude) and $U = V$. The matrix A has a repeated column vector in positions 1,3, linearly independent from column 2, hence $\operatorname{rank}(A) = 2$ and one of the eigenvalues must be 0, $\lambda_3 = 0$ with associated eigenvector $x_3 = (1, 0, -1)$. From observation of the structure of A (column vector 2 has no components along 1,3) one of the eigenvectors is $x_2 = (0, 1, 0)^T$ with eigenvalue $\lambda_2 = 1$. Completion of

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & -1 \end{pmatrix}$$

to form an orthogonal basis for $\mathbb{R}^3$ gives

$$X = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix}$$

with $\lambda_1 = 2$ the eigenvalue associated with $x_3 = (1,0,1)^T$. Bring X to normalized form

$$U = \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{pmatrix}$$

to obtain the SVD

$$A = U\Sigma U^T, \Sigma = \operatorname{diag}(2,1,0).$$

7. Revisit the midterm problem of computing A^{200} with

$$A = \begin{pmatrix} 1 & 0 & 1/2 & 1/2 \\ 0 & 1 & 1/2 & 1/2 \\ 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1/2 & 1/2 \end{pmatrix}.$$

Use the eigendecomposition of $A = X\Lambda X^{-1}$ to compute A^{200}.

Solution. Recognize the block structure

$$A = \begin{pmatrix} I & B \\ 0 & B \end{pmatrix}$$

and observe that

$$A\begin{pmatrix} I \\ 0 \end{pmatrix} = \begin{pmatrix} I \\ 0 \end{pmatrix}$$

hence two eigenvalue-eigenvector pairs are

$$\lambda_1 = 1, x_1 = e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \lambda_2 = 1, x_2 = e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

Row echelon reduction for $\lambda = 1$ gives

$$A - 1\cdot I = \begin{pmatrix} 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & -1/2 & 1/2 \\ 0 & 0 & 1/2 & -1/2 \end{pmatrix} \sim \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \sim \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & -2 \end{pmatrix} \sim \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The dimension of the null space of $A - 1\cdot I$, i.e., the geometric multiplicity of $\lambda = 1$ is 2 less than the algebraic multiplicity. The eigenvector matrix is therefore singular, and the matrix A cannot be diagonalized as $A = X\Lambda X^{-1}$. To compute A^{200} we have to proceed as in (cf.) the midterm solution and obtain

$$A^{200} = \begin{pmatrix} I & 200B \\ 0 & B \end{pmatrix}$$