MATH547 Final Examination

Fall 2015 Semester, December 11, 2015

Instructions. *Answer the following questions. Provide a motivation of your approach and the reasoning underlying successive steps in your solution. Write neatly and avoid erasures. Use scratch paper to sketch out your answer for yourself, and then transcribe your solution to the examination you turn in for grading. Illegible answers are not awarded any credit. Presentation of calculations without mention of the motivation and reasoning are not awarded any credit. The last, seventh question is optional and offered to enable raising your score on the midterm examination. Each complete, correct solution to an examination question is awarded 4 course grade points. Your primary goal should be to demonstrate understanding of course topics and skill in precise mathematical formulation and solution procedures.*

1. Consider the orthogonal matrix $\boldsymbol{Q} \in \mathbb{R}^{m \times m}$, $\boldsymbol{Q}^T \boldsymbol{Q} = \boldsymbol{I}$. Prove that $\det \boldsymbol{Q} = \pm 1$.

   Solution. Apply identities $\det(\boldsymbol{A}\boldsymbol{B}) = \det(\boldsymbol{A})\det(\boldsymbol{B})$, and $\det(\boldsymbol{A}^T) = \det(\boldsymbol{A})$ to $\boldsymbol{Q}^T\boldsymbol{Q} = \boldsymbol{I}$ to obtain

$$\det(\boldsymbol{Q}^T\boldsymbol{Q}) = \det(\boldsymbol{Q}^T)\det(\boldsymbol{Q}) = (\det(\boldsymbol{Q}))^2 = \det(\boldsymbol{I}) = 1 \Rightarrow \det(\boldsymbol{Q}) = \sqrt{1} = \pm 1. (q.e.d.)$$

2. Consider the matrix $\boldsymbol{V} \in \mathbb{R}^{3 \times 3}$ and cubic polynomial $p(t)$ defined by

$$\boldsymbol{V} = \begin{pmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{pmatrix}, p(t) = \det \begin{pmatrix} 1 & t & t^2 & t^3 \\ 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \end{pmatrix}.$$

   (a) Compute $\det \boldsymbol{V}$.

   Solution. Use determinant calculation rules shown above/below equal sign ($R1$, $C1$ denotes row 1, column 1 etc.) to obtain

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} \begin{matrix} R2 \leftarrow R2 - R1 \\ = \\ R3 \leftarrow R3 - R1 \end{matrix} = \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & (b-a)(b+a) \\ 0 & c-a & (c-a)(c+a) \end{vmatrix} \begin{matrix} \text{expand along } C1 \\ = \\ \text{factor } (b-a)(c-a) \end{matrix} =$$

$$(b-a)(c-a) \begin{vmatrix} 1 & b+a \\ 1 & c+a \end{vmatrix} = (b-a)(c-a)(c-b)$$

   (b) What are the roots of $p(t)$?

   Solution. Use rule that a determinant with repeated lines is null to obtain roots $\{a, b, c\}$.

3. Compute $\boldsymbol{A}^{16}$ for

$$\boldsymbol{A} = \begin{pmatrix} 4 & -3 \\ 2 & -1 \end{pmatrix}.$$

Solution. If $\boldsymbol{A}$ admits an eigendecomposition $\boldsymbol{A} = \boldsymbol{X}\boldsymbol{\Lambda}\boldsymbol{X}^{-1}$, then $\boldsymbol{A}^n = (\boldsymbol{X}\boldsymbol{\Lambda}\boldsymbol{X}^{-1})...(\boldsymbol{X}\boldsymbol{\Lambda}\boldsymbol{X}^{-1}) = \boldsymbol{X}\boldsymbol{\Lambda}^n\boldsymbol{X}^{-1}$. Determine if $\boldsymbol{A}$ admits an eigendecomposition (is diagonalizable) by computing the characteristic polynomial

$$p(\lambda) = \det(\boldsymbol{A} - \lambda\boldsymbol{I}) = \begin{vmatrix} 4-\lambda & -3 \\ 2 & -1-\lambda \end{vmatrix} = (\lambda - 1)(\lambda - 2).$$

Since $p(\lambda)$ has distinct roots $\lambda_1 = 1$, $\lambda_2 = 2$, the matrix admits an eigendecomposition. Determine eigenvectors by computing null space bases for $\boldsymbol{A} - \lambda_{1,2}\boldsymbol{I}$ by reduction to row echelon form

$$\begin{pmatrix} 4-\lambda_1 & -3 \\ 2 & -1-\lambda_1 \end{pmatrix} \sim \begin{pmatrix} 3 & -3 \\ 0 & 0 \end{pmatrix} \Rightarrow \boldsymbol{x}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 4-\lambda_2 & -3 \\ 2 & -1-\lambda_2 \end{pmatrix} \sim \begin{pmatrix} 2 & -3 \\ 0 & 0 \end{pmatrix} \Rightarrow \boldsymbol{x}_1 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

The inverse of the eigenvector matrix is computed by Jordan algorithm (row reduction to $\boldsymbol{I}$)

$$\begin{pmatrix} \boldsymbol{X} & \boldsymbol{I} \end{pmatrix} = \begin{pmatrix} 1 & 3 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & 1 & 0 \\ 0 & -1 & -1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -2 & 3 \\ 0 & 1 & 1 & -1 \end{pmatrix} \Rightarrow \boldsymbol{X}^{-1} = \begin{pmatrix} -2 & 3 \\ 1 & -1 \end{pmatrix}$$

The eigendecomposition of $\boldsymbol{A}$ is therefore

$$\boldsymbol{A} = \boldsymbol{X}\boldsymbol{\Lambda}\boldsymbol{X}^{-1} = \begin{pmatrix} 1 & 3 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} -2 & 3 \\ 1 & -1 \end{pmatrix},$$

leading to

$$\boldsymbol{A}^{16} = \boldsymbol{X}\boldsymbol{\Lambda}^{16}\boldsymbol{X}^{-1} = \begin{pmatrix} 1 & 3 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2^{16} \end{pmatrix} \begin{pmatrix} -2 & 3 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 3 \cdot 2^{16} - 2 & 3 \cdot (1 - 2^{16}) \\ 2(2^{16} - 1) & 3 - 2^{17} \end{pmatrix}.$$

4. Let λ be an eigenvalue of the nonsingular matrix $\boldsymbol{A} \in \mathbb{R}^{m \times m}$. Prove that λ^{-1} is an eigenvalue of $\boldsymbol{A}^{-1}$.

   Solution. First, show that $\boldsymbol{A}$ nonsingular implies $\lambda \neq 0$. Eigenvalues are roots of characteristic polynomial $p(\lambda) = \det(\boldsymbol{A} - \lambda\boldsymbol{I}) = 0$. If $\lambda = 0$ would be an eigenvalue, it would result that $\det(\boldsymbol{A}) = 0$, contradicting $\boldsymbol{A}$ nonsingular. Now, multiply eigenrelationship $\boldsymbol{A}\boldsymbol{x} = \lambda\boldsymbol{x}$ on left by $\boldsymbol{A}^{-1}$, and by λ^{-1} to obtain

$$\lambda^{-1}(\boldsymbol{A}^{-1}\boldsymbol{A})\boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{x} \Rightarrow \boldsymbol{A}^{-1}\boldsymbol{x} = \lambda^{-1}\boldsymbol{x}, \textit{q.e.d.}$$

5. Compute the singular value decomposition of

$$\boldsymbol{A} = \begin{pmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{pmatrix}.$$

   Solution 1 (long). The singular value decomposition is given by $\boldsymbol{A} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^T$, with $\boldsymbol{U} \in \mathbb{R}^{3\times 3}, \boldsymbol{V} \in \mathbb{R}^{2\times 2}$ orthogonal matrices and $\boldsymbol{\Sigma} \in \mathbb{R}_+^{3\times 2}$, $\boldsymbol{\Sigma} = \operatorname{diag}(\sigma_1, \sigma_2)$, $\sigma_1 \geqslant \sigma_2 \geqslant 0$. Form

$$\boldsymbol{A}^T\boldsymbol{A} = (\boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^T)^T\boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^T = \boldsymbol{V}\boldsymbol{\Sigma}^2\boldsymbol{V}^T,$$

and find that the squared singular values are eigenvalues of $\boldsymbol{A}^T\boldsymbol{A}$. Compute the characteristic polynomial

$$\det(\boldsymbol{A}^T\boldsymbol{A}-\lambda\boldsymbol{I})=\begin{pmatrix} 9-\lambda & -9 \\ -9 & -9-\lambda \end{pmatrix}=\lambda(\lambda-18)$$

so $\lambda_1=18\Rightarrow\sigma_1=\sqrt{18}=3\sqrt{2}$, $\lambda_2=0\Rightarrow\sigma_2=0$. Deduce that $\operatorname{rank}(\boldsymbol{A})=r=1$. Compute eigenvectors of $\boldsymbol{A}^T\boldsymbol{A}$ (right singular vectors of $\boldsymbol{A}$) by determining bases for null spaces $\boldsymbol{A}^T\boldsymbol{A}-\lambda_{1,2}\boldsymbol{I}$.

$$\boldsymbol{A}^T\boldsymbol{A}-18\cdot\boldsymbol{I}=\begin{pmatrix} -9 & -9 \\ -9 & -9 \end{pmatrix}\Rightarrow\boldsymbol{v}_1=\begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

$$\boldsymbol{A}^T\boldsymbol{A}-0\cdot\boldsymbol{I}=\begin{pmatrix} 9 & -9 \\ -9 & 9 \end{pmatrix}\Rightarrow\boldsymbol{v}_2=\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

so

$$\boldsymbol{V}=(\,\boldsymbol{v}_1\;\;\boldsymbol{v}_2\,)=\begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}.$$

To find $\boldsymbol{U}$, compute

$$\boldsymbol{A}\boldsymbol{A}^T=\begin{pmatrix} 2 & -4 & 4 \\ -4 & 8 & -8 \\ 4 & -8 & 8 \end{pmatrix}$$

and find eigenvector associated with $\lambda_1=18$ by row echelon reduction

$$\boldsymbol{A}\boldsymbol{A}^T-18\cdot\boldsymbol{I}=\begin{pmatrix} -16 & -4 & 4 \\ -4 & -10 & -8 \\ 4 & -8 & -10 \end{pmatrix}\sim\begin{pmatrix} 4 & 1 & -1 \\ -4 & -10 & -8 \\ 4 & -8 & -10 \end{pmatrix}\sim\begin{pmatrix} 4 & 1 & -1 \\ 0 & -9 & -9 \\ 0 & 0 & 0 \end{pmatrix}\Rightarrow\boldsymbol{u}_1=\frac{1}{3}\begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix}.$$

The other two left singular vectors $\boldsymbol{u}_2,\boldsymbol{u}_3$ are orthogonal to $\boldsymbol{u}_1$. By observation

$$\boldsymbol{u}_2=\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix},\boldsymbol{u}_3=\frac{1}{\sqrt{18}}\begin{pmatrix} 4 \\ 1 \\ -1 \end{pmatrix},$$

leading to

$$\boldsymbol{A}=\boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^T=\begin{pmatrix} -1/3 & 0 & 4/\sqrt{18} \\ 2/3 & 1/\sqrt{2} & 1/\sqrt{18} \\ -2/3 & 1/\sqrt{2} & -1/\sqrt{18} \end{pmatrix}\begin{pmatrix} \sqrt{18} & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}.$$

Solution 2 (concise). The singular value decomposition is given by $\boldsymbol{A}=\boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^T$, with $\boldsymbol{U}\in\mathbb{R}^{3\times 3}$, $\boldsymbol{V}\in\mathbb{R}^{2\times 2}$ orthogonal matrices and $\boldsymbol{\Sigma}\in\mathbb{R}_+^{3\times 2}$, $\boldsymbol{\Sigma}=\operatorname{diag}(\sigma_1,\sigma_2)$, $\sigma_1\geqslant\sigma_2\geqslant 0$. Since in $\boldsymbol{A}=(\,\boldsymbol{a}_1\;\;\boldsymbol{a}_2\,)$, $\boldsymbol{a}_1=-\boldsymbol{a}_2$ deduce $r=\operatorname{rank}(\boldsymbol{A})=1$, $\sigma_2=0$, and an orthonormal basis for $C(\boldsymbol{A})$ is $\boldsymbol{u}_1=\boldsymbol{a}_1/\|\boldsymbol{a}_1\|$ and SVD must be of form

$$\boldsymbol{A}=\boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^T=(\,\boldsymbol{u}_1\;\;\boldsymbol{u}_2\;\;\boldsymbol{u}_3\,)\begin{pmatrix} \sigma_1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}(\,\boldsymbol{v}_1\;\;\boldsymbol{v}_2\,)^T=\sigma_1\boldsymbol{u}_1\boldsymbol{v}_1^T.$$

Form $\boldsymbol{A}^T\boldsymbol{A} = (\boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^T)^T\boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^T = \boldsymbol{V}\boldsymbol{\Sigma}^2\boldsymbol{V}^T$, and find that the squared singular values are eigenvalues of $\boldsymbol{A}^T\boldsymbol{A}$. Compute the characteristic polynomial

$$\det(\boldsymbol{A}^T\boldsymbol{A} - \lambda\boldsymbol{I}) = \begin{pmatrix} 9-\lambda & -9 \\ -9 & -9-\lambda \end{pmatrix} = \lambda(\lambda - 18)$$

so $\lambda_1 = 18 \Rightarrow \sigma_1 = \sqrt{18} = 3\sqrt{2}$. Write

$$\boldsymbol{A} = \begin{pmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{pmatrix} = 3\sqrt{2} \cdot \frac{1}{3}\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}\begin{pmatrix} v_{11} & v_{12} \end{pmatrix}$$

and by inspection deduce $\boldsymbol{v}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \end{pmatrix}$. Now complete $\boldsymbol{U}, \boldsymbol{V}$ to form bases for $\mathbb{R}^3, \mathbb{R}^2$ respectively.

6. Consider the ordinary differential equation system $\boldsymbol{x}'(t) = \boldsymbol{A}\boldsymbol{x}(t)$ with $\boldsymbol{x}(t): \mathbb{R} \to \mathbb{R}^2$, $\boldsymbol{A} \in \mathbb{R}^{2\times 2}$ with eigenvalues $\lambda_1 = -3$, $\lambda_2 = -1$, and corresponding eigenvectors

$$\boldsymbol{v}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \boldsymbol{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Compute solution at $t = 1$, $\boldsymbol{x}(1)$ given the initial condition at $t = 0$,

$$\boldsymbol{x}(0) = \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

Solution. $\boldsymbol{A}$ has distinct eigenvalues, hence is diagonalizable, $\boldsymbol{A} = \boldsymbol{V}\boldsymbol{\Lambda}\boldsymbol{V}^{-1}$, such that the ODE system becomes $\boldsymbol{x}'(t) = \boldsymbol{V}\boldsymbol{\Lambda}\boldsymbol{V}^{-1}\boldsymbol{x}(t)$. Let $\boldsymbol{y}(t) = \boldsymbol{V}^{-1}\boldsymbol{x}(t)$. Compute $\boldsymbol{V}^{-1}$

$$\begin{pmatrix} -1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & -1 & 0 \\ 0 & 2 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & -1 & 0 \\ 0 & 1 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \Rightarrow \boldsymbol{V}^{-1} = \frac{1}{2}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$$

and find

$$\boldsymbol{y}(0) = \frac{1}{2}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}\boldsymbol{x}(0) = \frac{1}{2}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ 5 \end{pmatrix}.$$

Expressed in terms of $\boldsymbol{y}(t)$ the ODE system becomes $\boldsymbol{y}'(t) = \boldsymbol{\Lambda}\boldsymbol{y}(t)$ with solution

$$\boldsymbol{y}(t) = \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix}\boldsymbol{y}(0) \Rightarrow \boldsymbol{y}(1) = \frac{1}{2}\begin{pmatrix} e^{-3} & 0 \\ 0 & e^{-1} \end{pmatrix}\begin{pmatrix} 1 \\ 5 \end{pmatrix} = \frac{1}{2e^3}\begin{pmatrix} 1 \\ 5e^2 \end{pmatrix},$$

whence

$$\boldsymbol{x}(1) = \boldsymbol{V}\boldsymbol{y}(1) = \frac{1}{2e^3}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 5e^2 \end{pmatrix} = \frac{1}{2e^3}\begin{pmatrix} -1+5e^2 \\ 1+5e^2 \end{pmatrix}.$$

7. Consider $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ with the property that $\boldsymbol{A}^T\boldsymbol{A}$ is nonsingular. Prove that the columns of $\boldsymbol{A}$ are linearly independent.

Solution. By definition, $\boldsymbol{A}$ has linearly independent columns if the only solution of $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}$, is $\boldsymbol{x} = \boldsymbol{0}$. Multiply $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}$ on the left by $\boldsymbol{A}^T$, to obtain $(\boldsymbol{A}^T\boldsymbol{A})\,\boldsymbol{x} = \boldsymbol{0}$. Since $\boldsymbol{A}^T\boldsymbol{A}$ is non-singular the only solution is indeed $\boldsymbol{x} = 0$ (q.e.d.)