MATH547 Final Examination Solution

Spring 2017 Semester, May 4, 2017

Instructions. *Answer the following questions. Provide a motivation of your approach and the reasoning underlying successive steps in your solution. Write neatly and avoid erasures. Use scratch paper to sketch out your answer for yourself, and then transcribe your solution to the examination you turn in for grading. Illegible answers are not awarded any credit. Presentation of calculations without mention of the motivation and reasoning are not awarded any credit. Each complete, correct solution to an examination question is awarded 4 course grade points. The last, seventh question is optional and offered to enable raising your score on the midterm examination. Your primary goal should be to demonstrate understanding of course topics and skill in precise mathematical formulation and solution procedures.*

1. Consider the matrix $\boldsymbol{Q} = (\ \boldsymbol{q}_1 \ \ \boldsymbol{q}_2 \ \ \ldots \ \ \boldsymbol{q}_n \ ) \in \mathbb{R}^{m \times n}$ with orthonormal columns, $\boldsymbol{q}_i^T \boldsymbol{q}_j = \delta_{ij}$. Determine the eigenvalues and eigenvectors of the projection matrix $\boldsymbol{P} = \boldsymbol{Q}\boldsymbol{Q}^T$ onto $C(\boldsymbol{Q})$.

Solution. For any $\boldsymbol{v} \in C(\boldsymbol{Q})$, $\boldsymbol{P}\boldsymbol{v} = \boldsymbol{v}$, hence $\lambda = 1$ is an eigenvalue of algebraic multiplicity n (dimension of $C(\boldsymbol{Q})$), and geometric multiplicity n, with eigenvectors $\boldsymbol{q}_1, \boldsymbol{q}_2, \ldots, \boldsymbol{q}_n$. For any $\boldsymbol{w} \in N(\boldsymbol{Q}^T)$, $\boldsymbol{P}\boldsymbol{w} = 0$, hence $\lambda = 0$ is an eigenvalue of algebraic and geometric multiplicities $m - n$. Eigenvectors are the $m - n$ basis vectors of $N(\boldsymbol{Q}^T)$.

2. It is known that $\boldsymbol{A} \in \mathbb{R}^{3 \times 3}$ has eigenvalues $\lambda_1 = 0$, $\lambda_2 = 1$, $\lambda_3 = 2$.

   a) Does the linear system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ ($\boldsymbol{x}, \boldsymbol{b} \in \mathbb{R}^3$) have a solution?

   Solution. By FTLA, $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ has a solution if $\boldsymbol{b} \in C(\boldsymbol{A})$. Since $\boldsymbol{A}$ has distinct eigenvalues, it is diagonalizable, $\boldsymbol{A} = \boldsymbol{X}\boldsymbol{\Lambda}\boldsymbol{X}^{-1}$, and system can be rewritten as $\boldsymbol{\Lambda}\boldsymbol{y} = \boldsymbol{c}$, with $\boldsymbol{y} = \boldsymbol{X}^{-1}\boldsymbol{x}$, $\boldsymbol{c} = \boldsymbol{X}^{-1}\boldsymbol{b}$. System will have solution iff $c_1 = 0$ since $\lambda_1 = 0$.

   b) Is the solution to the linear system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ unique?

   Solution. Since $\det(\boldsymbol{A}) = \lambda_1\lambda_2\lambda_3 = 0$, $\boldsymbol{A}$ is singular. If $\boldsymbol{b} \notin C(\boldsymbol{A})$ there is no solution, and if $\boldsymbol{b} \in C(\boldsymbol{A})$ there are infinitely many solutions (one-parameter family of solutions).

   c) What are the eigenvalues of $(\boldsymbol{I} + \boldsymbol{A}^2)^{-1}$?

   Solution. From $(\boldsymbol{I} + \boldsymbol{A}^2)^{-1}\boldsymbol{y} = \mu\boldsymbol{y}$, obtain $(\boldsymbol{I} + \boldsymbol{A}^2)\boldsymbol{y} = \frac{1}{\mu}\boldsymbol{y}$. Assume $\boldsymbol{y}$ is an eigenvector of $\boldsymbol{A}$, $\boldsymbol{A}\boldsymbol{y} = \lambda\boldsymbol{y}$. Then, $(\boldsymbol{I} + \boldsymbol{A}^2)\boldsymbol{y} = (1 + \lambda^2)\boldsymbol{y}$, hence $\mu = 1/(1 + \lambda^2)$, and the eigenvalues of $(\boldsymbol{I} + \boldsymbol{A}^2)^{-1}$ are $\mu_1 = 1$, $\mu_2 = 1/2$, $\mu_3 = 1/5$.

3. Determine $\boldsymbol{A}^{31}$ with

$$\boldsymbol{A} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

Solution. $\boldsymbol{A}$ is symmetric, hence unitarily diagonalizable, $\boldsymbol{A}=\boldsymbol{Q}\boldsymbol{\Lambda}\boldsymbol{Q}^T$, with $\boldsymbol{Q}$ an orthogonal matrix, $\boldsymbol{Q}\boldsymbol{Q}^T=\boldsymbol{Q}^T\boldsymbol{Q}=\boldsymbol{I}$, and $\boldsymbol{A}^{31}=\boldsymbol{Q}\boldsymbol{\Lambda}^{31}\boldsymbol{Q}^T$. Compute

$$p(\lambda)=\det(\boldsymbol{A}-\lambda\boldsymbol{I})=\lambda^2-4\lambda+3=(\lambda-1)(\lambda-3),$$

to find eigenvalues $\lambda_1=1,\lambda_2=3$. Associated eigenvectors are within null spaces $N(\boldsymbol{A}-\lambda_i\boldsymbol{I})$

$$\lambda_1=1\colon \boldsymbol{A}-\lambda_1\boldsymbol{I}=\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\sim\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}\Rightarrow \boldsymbol{q}_1=\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

$$\lambda_2=3\colon \boldsymbol{A}-\lambda_2\boldsymbol{I}=\begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}\sim\begin{pmatrix} -1 & -1 \\ 0 & 0 \end{pmatrix}\Rightarrow \boldsymbol{q}_2=\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Compute

$$\boldsymbol{A}^{31}=\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 3^{31} \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}=\frac{1}{2}\begin{pmatrix} 1+3^{31} & 1-3^{31} \\ 1-3^{31} & 1+3^{31} \end{pmatrix}.$$

4. Prove that the eigenvectors of a real, symmetric matrix with distinct eigenvalues are orthogonal.

Solution. Consider $\boldsymbol{A}\in\mathbb{R}^{m\times m}$, $\boldsymbol{A}=\boldsymbol{A}^T$, and $\boldsymbol{A}\boldsymbol{x}_i=\lambda_i\boldsymbol{x}_i$, $\boldsymbol{A}\boldsymbol{x}_j=\lambda_j\boldsymbol{x}_j$, with $\lambda_i\neq\lambda_j$. Transposition of eigenvalue equalities gives $\boldsymbol{x}_i^T\boldsymbol{A}^T=\boldsymbol{x}_i^T\boldsymbol{A}=\lambda_i\boldsymbol{x}_i^T$, $\boldsymbol{x}_j^T\boldsymbol{A}^T=\boldsymbol{x}_j^T\boldsymbol{A}=\lambda_j\boldsymbol{x}_j^T$, using $\boldsymbol{A}=\boldsymbol{A}^T$. Compute $\boldsymbol{x}_i^T\boldsymbol{A}\boldsymbol{x}_j-\boldsymbol{x}_j^T\boldsymbol{A}\boldsymbol{x}_i=0=(\lambda_i-\lambda_j)\boldsymbol{x}_i^T\boldsymbol{x}_j\Rightarrow\boldsymbol{x}_i^T\boldsymbol{x}_j=0$, hence $\boldsymbol{x}_i\perp\boldsymbol{x}_j$.

5. Consider $\boldsymbol{A}=(\begin{array}{cccc}\boldsymbol{a}_1 & \boldsymbol{a}_2 & \dots & \boldsymbol{a}_n\end{array})\in\mathbb{R}^{m\times n}$ with orthogonal columns, $i\neq j\Rightarrow\boldsymbol{a}_i^T\boldsymbol{a}_j=\delta_{ij}$. Determine the singular value decomposition (SVD) of $\boldsymbol{A}$, i.e., $\boldsymbol{A}=\boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^T$.

Solution. Let $\sigma_i=\|\boldsymbol{a}_{p(i)}\|$ with $p(i)$ a permutation such that $\sigma_1\geqslant\sigma_2\geqslant\dots\geqslant\sigma_n$, and let $\boldsymbol{u}_i=\boldsymbol{a}_i/\|\boldsymbol{a}_i\|\in\mathbb{R}^m$ for $i=1,\dots,n$. The SVD is

$$\boldsymbol{A}=(\begin{array}{cccccc}\boldsymbol{u}_{p(1)} & \dots & \boldsymbol{u}_{p(n)} & \boldsymbol{u}_{n+1} & \dots & \boldsymbol{u}_m\end{array})\begin{pmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_n \\ & 0 & \end{pmatrix}\boldsymbol{I}_n.$$

6. Compute the singular value decomposition of

$$\boldsymbol{A}=\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

Solution. The SVD is $\boldsymbol{A}=\boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^T$ with $\boldsymbol{U}\in\mathbb{R}^{m\times m}, \boldsymbol{V}\in\mathbb{R}^{n\times n}$ the singular vector orthogonal matrices and $\boldsymbol{\Sigma}\in\mathbb{R}_+^{m\times n}$ diagonal with entries $\sigma_1\geqslant\ldots\geqslant\sigma_r>\sigma_{r+1}=\ldots\sigma_n=0$ (the singular values). From $\boldsymbol{A}^T\boldsymbol{A}=\boldsymbol{V}\boldsymbol{\Sigma}^T\boldsymbol{\Sigma}\boldsymbol{V}^T$ deduce that $\sigma_i^2=\lambda_i$, with λ_i eigenvalues from $(\boldsymbol{A}^T\boldsymbol{A})\boldsymbol{u}_i=\lambda_i\boldsymbol{u}_i$. Compute

$$p(\lambda)=\det(\boldsymbol{A}^T\boldsymbol{A}-\lambda\boldsymbol{I})=\begin{vmatrix} 4-\lambda & 0 & 0 \\ 0 & 5 & 4 \\ 0 & 4 & 5 \end{vmatrix}=(4-\lambda)(\lambda-1)(\lambda-9),$$

hence $\sigma_1=3$, $\sigma_2=2$, $\sigma_1=1$. Find eigenvectors:

$$\boldsymbol{A}^T\boldsymbol{A}-\lambda_1\boldsymbol{I}=\begin{pmatrix} -5 & 0 & 0 \\ 0 & -4 & 4 \\ 0 & 4 & -4 \end{pmatrix}\sim\begin{pmatrix} -5 & 0 & 0 \\ 0 & -4 & 4 \\ 0 & 0 & 0 \end{pmatrix}\Rightarrow\boldsymbol{v}_1=\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

Repeat to find the SVD

$$\boldsymbol{A}=\begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 3 & & \\ & 2 & \\ & & 1 \\ & & \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix}.$$

7. Find bases for the four fundamental subspaces $C(\boldsymbol{A})$, $N(\boldsymbol{A}^T)$, $C(\boldsymbol{A}^T)$, $N(\boldsymbol{A})$ associated with the matrix

$$\boldsymbol{A}=\begin{pmatrix} 1 & -3 & 0 \\ 2 & -6 & 4 \\ -3 & 9 & 1 \end{pmatrix}.$$

Specify the rank and nullity of $\boldsymbol{A}$.

Solution. $\boldsymbol{A}=(\ \boldsymbol{a}_1\ \ \boldsymbol{a}_2\ \ \boldsymbol{a}_3\ )\in\mathbb{R}^{m\times n}$, $m=n=3$. Since $\boldsymbol{a}_2=-3\boldsymbol{a}_1$, and $\alpha_1\boldsymbol{a}_1+\alpha_3\boldsymbol{a}_3=\boldsymbol{0}\Rightarrow\alpha_1=\alpha_3=0$, deduce $r=\operatorname{rank}(\boldsymbol{A})=2$, $\operatorname{nullity}(\boldsymbol{A})=1$, and

$$C(\boldsymbol{A})=\operatorname{span}\left\{\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix},\begin{pmatrix} 0 \\ 4 \\ 1 \end{pmatrix}\right\}.$$

Use cross product $\boldsymbol{a}_1\times\boldsymbol{a}_3$ to find basis for $N(\boldsymbol{A}^T)=\operatorname{span}\{(\ 14\ \ -1\ \ 4\ )^T\}$.

$$N(\boldsymbol{A})=\operatorname{span}\left\{\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}\right\},C(\boldsymbol{A}^T)=\operatorname{span}\left\{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},\begin{pmatrix} 1 \\ -3 \\ 0 \end{pmatrix}\right\}.$$