MATH547 Midterm Examination

Fall 2014 Semester, October 14, 2014

Instructions. *Answer the following questions. Provide a motivation of your approach and the reasoning underlying successive steps in your solution. Write neatly and avoid erasures. Use scratch paper to sketch out your answer for yourself, and then transcribe your solution to the examination you turn in for grading. Illegible answers are not awarded any credit. Presentation of calculations without mention of the motivation and reasoning are not awarded any credit. Each complete, correct solution to an examination question is awarded 4 course grade points. Your primary goal should be to demonstrate understanding of course topics and skill in precise mathematical formulation and solution procedures.*

1. Consider a vector $b \in \mathbb{R}^{3\times 1}$ with components $b_1 = 1$, $b_2 = 2$, $b_3 = -1$ in the canonical basis

$$\{e_1, e_2, e_3\} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Let x_1, x_2, x_3 denote the components of b with respect to the basis

$$\{A_1, A_2, A_3\} = \left\{ \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

Compute $x = (\, x_1 \;\; x_2 \;\; x_3 \,)^T \in \mathbb{R}^3$.

Solution. The problem asks for the linear combination of columns of A that gives b,

$$Ax = b$$

$$\begin{pmatrix} 1 & 0 & 2 \\ -1 & 1 & 1 \\ 2 & 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

Form the extended matrix and carry out reduction to upper triangular form

$$\left(\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ -1 & 1 & 1 & 2 \\ 2 & 2 & 1 & -1 \end{array} \right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & 3 & 3 \\ 0 & 2 & -3 & -3 \end{array} \right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & -9 & -9 \end{array} \right)$$

Back substitution

$$\left(\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & -9 & -9 \end{array} \right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array} \right)$$

with solution

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

Verify

$$\begin{pmatrix} 1 & 0 & 2 \\ -1 & 1 & 1 \\ 2 & 2 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \checkmark$$

2. Let

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \\ 3 & 1 \end{pmatrix}$$

a) Find bases for the four fundamental subspaces $C(A)$, $C(A^T)$, $N(A)$, $N(A^T)$.

b) Compute the orthogonal projection of the vector $b = (\begin{array}{ccc} 1 & 1 & 1 \end{array})^T$ onto $C(A)$.

Solution. a) Carry out reduction to reduced row echelon form

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \\ 3 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 \\ 0 & -3 \\ 0 & -5 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 \\ 0 & -3 \\ 0 & -5 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

From the above, $\operatorname{rank}(A) = \dim C(A) = \dim C(A^T) = 2$, and both columns of A are pivot columns. Since $A \in \mathbb{R}^{3\times 2}$

$$\dim C(A) + \dim N(A^T) = 3 \Rightarrow \dim N(A^T) = 1,$$

$$\dim C(A^T) + \dim N(A) = 2 \Rightarrow \dim N(A) = 0.$$

A basis for the row space is

$$C(A^T) = \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle,$$

and $N(A) = \{0\}$.

The columns of A are linearly independent, so a basis for $C(A) = \{Ax \,|\, x \in \mathbb{R}^2\}$ is $\{A_1, A_2\}$,

$$C(A) = \left\langle \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \right\rangle.$$

From definition of $N(A^T)=\{y\in\mathbb{R}^3|\, A^T y=0\in\mathbb{R}^2\}$, obtain system

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

or, in traditional form,

$$\begin{cases} y_1+2y_2+3y_3=0 \\ 2y_1+y_2+y_3=0 \end{cases}.$$

Set $y_3=-\alpha$ as a free variable and

$$\left(\begin{array}{cc|c} 1 & 2 & 3\alpha \\ 2 & 1 & \alpha \end{array}\right)\sim\left(\begin{array}{cc|c} 1 & 2 & 3\alpha \\ 0 & -3 & -5\alpha \end{array}\right)$$

so the solution is

$$y=\alpha\begin{pmatrix} -1/3 \\ 5/3 \\ -1 \end{pmatrix}$$

so

$$N(A^T)=\langle w\rangle=\left\langle \frac{1}{\sqrt{35}}\begin{pmatrix} -1 \\ 5 \\ -3 \end{pmatrix}\right\rangle$$

b) The orthogonal projection $c\in C(A)$ can be found in one of three ways:

i. $c=P_Q b$, $P_Q=QQ^T$, $Q\in\mathbb{R}^{3\times 2}$ obtain by orthonormalizaing columns of A

ii. $c=P_A b$, $P=A(A^TA)^{-1}A^T$

iii. Noticing that $c\in C(A)$ means $b-c\in N(A^T)$, hence $c=b-(w^Tb)w$ is the desired projection.

Method (iii) is the fastest

$$c=\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}-\frac{1}{35}\begin{pmatrix} -1 & 5 & -3 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\begin{pmatrix} -1 \\ 5 \\ -3 \end{pmatrix}=\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}-\frac{1}{35}\begin{pmatrix} -1 \\ 5 \\ -3 \end{pmatrix}=\begin{pmatrix} 36/35 \\ 6/7 \\ 38/35 \end{pmatrix}$$

For comparison, here are the extensive calculations for procedure (i),

$$c=P_Q b$$

with $P_Q = QQ^T$, and $Q = (Q_1\ Q_2)$ an orthogonal matrix obtained from a basis for $C(A)$. Apply Gram-Schmidt to obtain

$$Q_1 = A_1/\|A_1\| = \frac{1}{\sqrt{14}}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$w = A_2 - (Q_1^T A_2)Q_1 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} - \frac{7}{\sqrt{14}} \cdot \frac{1}{\sqrt{14}}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 3/2 \\ 0 \\ -1/2 \end{pmatrix}$$

$$Q_2 = \frac{\sqrt{10}}{5}\begin{pmatrix} 3/2 \\ 0 \\ -1/2 \end{pmatrix}$$

$$P_Q = QQ^T = \begin{pmatrix} 1/\sqrt{14} & 3\sqrt{10}/10 \\ 2/\sqrt{14} & 0 \\ 3/\sqrt{14} & -\sqrt{10}/10 \end{pmatrix}\begin{pmatrix} 1/\sqrt{14} & 2/\sqrt{14} & 3/\sqrt{14} \\ 3\sqrt{10}/10 & 0 & -\sqrt{10}/10 \end{pmatrix}$$

$$P_Q = \begin{pmatrix} 34/35 & 1/7 & -3/35 \\ 1/7 & 2/7 & 3/7 \\ -3/35 & 3/7 & 26/35 \end{pmatrix}$$

And the desired projection is

$$c = \begin{pmatrix} 34/35 & 1/7 & -3/35 \\ 1/7 & 2/7 & 3/7 \\ -3/35 & 3/7 & 26/35 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 36/35 \\ 6/7 \\ 38/35 \end{pmatrix}$$

3. Compute A^{200} with

$$A = \begin{pmatrix} 1 & 0 & 1/2 & 1/2 \\ 0 & 1 & 1/2 & 1/2 \\ 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1/2 & 1/2 \end{pmatrix}.$$

Solution. The matrix A has the following block structure

$$A = \begin{pmatrix} I & B \\ 0 & B \end{pmatrix}, I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, 0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$$

Compute A^2

$$A^2 = \begin{pmatrix} I & B \\ 0 & B \end{pmatrix}\begin{pmatrix} I & B \\ 0 & B \end{pmatrix} = \begin{pmatrix} I & 2B \\ 0 & B^2 \end{pmatrix}$$

Note that

$$B^2 = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix},$$

so

$$A^2 = \begin{pmatrix} I & 2B \\ 0 & B \end{pmatrix}$$

Now, compute

$$A^3 = \begin{pmatrix} I & 2B \\ 0 & B \end{pmatrix} \begin{pmatrix} I & B \\ 0 & B \end{pmatrix} = \begin{pmatrix} I & 3B \\ 0 & B^2 \end{pmatrix} = \begin{pmatrix} I & 3B \\ 0 & B \end{pmatrix}.$$

The pattern is

$$A^n = \begin{pmatrix} I & nB \\ 0 & B \end{pmatrix},$$

so

$$A^{200} = \begin{pmatrix} 1 & 0 & 100 & 100 \\ 0 & 1 & 100 & 100 \\ 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1/2 & 1/2 \end{pmatrix}.$$