MATH547 Midterm Examination

Spring 2016 Semester, March 11, 2016

Instructions. *Answer the following questions. Provide a motivation of your approach and the reasoning underlying successive steps in your solution. Write neatly and avoid erasures. Use scratch paper to sketch out your answer for yourself, and then transcribe your solution to the examination you turn in for grading. Illegible answers are not awarded any credit. Presentation of calculations without mention of the motivation and reasoning are not awarded any credit. Each complete, correct solution to an examination question is awarded 4 course grade points. Your primary goal should be to demonstrate understanding of course topics and skill in precise mathematical formulation and solution procedures.*

1. Determine orthonormal bases for $C(\boldsymbol{A})$ (column space of A) and $N(\boldsymbol{A}^T)$ (left null space of A) with

$$\boldsymbol{A} = \begin{pmatrix} 0 & 1 & 3 \\ 1 & -1 & -1 \\ 1 & 0 & 2 \end{pmatrix}.$$

Solution. By fundamental theorem of linear algebra (FTLA), for $\boldsymbol{A} \in \mathbb{R}^{m\times n}$, $\mathbb{R}^n \xrightarrow{\boldsymbol{A}} \mathbb{R}^m$, $C(\boldsymbol{A}) \oplus N(\boldsymbol{A}^T) = \mathbb{R}^m$, $C(\boldsymbol{A}) \perp N(\boldsymbol{A}^T)$, with $m = n = 3$. Solve system $\boldsymbol{A}^T\boldsymbol{y} = \boldsymbol{0}$ by Gaussian elimination

$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ 3 & -1 & 2 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 3 & -1 & 2 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & 2 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \boldsymbol{y} = \alpha \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}.$$

Since reduction to row echelon form identified 2 pivot columns, deduce that $\operatorname{rank}(\boldsymbol{A}^T) = \dim C(\boldsymbol{A}^T) = 2$, and since dimension of row space of a matrix equals that of column space, $\dim C(\boldsymbol{A}) = 2$, $\dim N(\boldsymbol{A}^T) = 1$. From above solution, an orthonormal basis for $N(\boldsymbol{A}^T)$ is

$$N(\boldsymbol{A}^T) = \operatorname{span}\{\boldsymbol{v}_1\}, \boldsymbol{v}_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}, \|\boldsymbol{v}_1\| = 1.$$

Observe that

$$\boldsymbol{a}_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \boldsymbol{a}_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

are both orthogonal to $\boldsymbol{v}_1$ ($\boldsymbol{a}_1^T\boldsymbol{v}_1 = 0$, $\boldsymbol{a}_2^T\boldsymbol{v}_1 = 0$), and linearly independent, since rref gives

$$\begin{pmatrix} \boldsymbol{a}_2 & \boldsymbol{a}_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \\ 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}$$

with 2 pivot columns. By FTLA, $C(\boldsymbol{A}) = \operatorname{span}\{\boldsymbol{a}_1, \boldsymbol{a}_2\}$. Find an orthonormal basis $\{\boldsymbol{u}_1, \boldsymbol{u}_2\}$ by Gram-Schmidt algorithm:

$$\boldsymbol{u}_1 = \frac{1}{\|\boldsymbol{a}_1\|}\boldsymbol{a}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$\boldsymbol{w} = \boldsymbol{a}_2 - (\boldsymbol{u}_1^T\boldsymbol{a}_2)\boldsymbol{u}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} - \frac{(-1)}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

$$\boldsymbol{u}_2 = \frac{1}{\|\boldsymbol{w}\|}\boldsymbol{w} = \sqrt{\frac{2}{3}}\begin{pmatrix} 1 \\ -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix}.$$

2. Which of the following are vector subspaces of $\mathbb{R}^3$ (provide justification for your answer)

   a) $\mathcal{S} = \{(x, y, z)^T | x + y + z + 1 = 0\}$.

      Solution. Since $\mathbf{0} \notin \mathcal{S}$, $\mathcal{S}$ is not a vector subspace of $\mathbb{R}^3$.

   b) $\mathcal{S} = \{(x, y, z)^T | x \geqslant y \geqslant z \geqslant 0\}$.

      Solution. Since $-\boldsymbol{u} \notin \mathcal{S}$ for $\boldsymbol{u} = (3, 2, 1) \in \mathcal{S}$, deduce that $\mathcal{S}$ is not a vector space, and hence cannot be a vector subspace of $\mathbb{R}^3$.

3. Find the (minimal) distance from the point $\boldsymbol{b} = (1, 2, -1)^T$ to the plane $x - 2y + z = 0$.

Solution. (Algebraic) The equation for the plane can be written as $\boldsymbol{A}\boldsymbol{r} = \mathbf{0}$, with

$$\boldsymbol{A} = \begin{pmatrix} 1 & -2 & 1 \end{pmatrix} \in \mathbb{R}^{1\times 3}, \boldsymbol{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Points within the plane are elements of $N(\boldsymbol{A})$. The matrix $\boldsymbol{A}$ is in reduced row echelon form with 1 pivot column, hence $\operatorname{rank}(\boldsymbol{A}) = 1 = \dim C(\boldsymbol{A}^T) = \dim C(\boldsymbol{A})$. Since $C(\boldsymbol{A}^T) \oplus N(\boldsymbol{A}) = \mathbb{R}^3$, and $C(\boldsymbol{A}^T) \perp N(\boldsymbol{A})$, the vector $\boldsymbol{b}$ can be decomposed as $\boldsymbol{b} = \boldsymbol{c} + \boldsymbol{d}$, $\boldsymbol{c} \in N(\boldsymbol{A}), d \in C(\boldsymbol{A}^T)$. An orthonormal basis for $C(\boldsymbol{A}^T)$ is given by the single vector

$$\boldsymbol{n} = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}.$$

The distance from $\boldsymbol{b}$ to the plane is the norm of projection of $\boldsymbol{b}$ onto $C(\boldsymbol{A}^T)$

$$\boldsymbol{d} = \boldsymbol{P_n b} = (\boldsymbol{n}\boldsymbol{n}^T)\boldsymbol{b} = (\boldsymbol{n}^T\boldsymbol{b})\boldsymbol{n} \Rightarrow d = \|\boldsymbol{d}\| = \|(\boldsymbol{n}^T\boldsymbol{b})\boldsymbol{n}\| = |(\boldsymbol{n}^T\boldsymbol{b})|\|\boldsymbol{n}\| = |(\boldsymbol{n}^T\boldsymbol{b})|,$$

$$d = \left| \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & -2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \right| = \frac{|1-4-1|}{\sqrt{6}} = \frac{4}{\sqrt{6}} = \frac{2}{3}\sqrt{6}.$$

(Geometric) In vector form the plane is expressed as

$$\boldsymbol{r}^T\boldsymbol{n} = 0, \text{with } \boldsymbol{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \boldsymbol{n} = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix},$$

with $\boldsymbol{n}$ the unit normal vector to the plane. The projection of $\boldsymbol{b}$ onto span$\{\boldsymbol{n}\}$ is

$$\boldsymbol{d} = \boldsymbol{P_n b} = (\boldsymbol{n}\boldsymbol{n}^T)\boldsymbol{b} = \boldsymbol{n}(\boldsymbol{n}^T\boldsymbol{b}) = (\boldsymbol{n}^T\boldsymbol{b})\boldsymbol{n} = \frac{-4}{\sqrt{6}} \cdot \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = -\frac{2}{3} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}.$$

The minimal distance is $\|\boldsymbol{d}\| = \frac{2}{3}\sqrt{6}$.