MATH547 Midterm Examination

Spring 2017 Semester, March 10, 2017

Instructions. *Answer the following questions. Provide a motivation of your approach and the reasoning underlying successive steps in your solution. Write neatly and avoid erasures. Use scratch paper to sketch out your answer for yourself, and then transcribe your solution to the examination you turn in for grading. Illegible answers are not awarded any credit. Presentation of calculations without mention of the motivation and reasoning are not awarded any credit. Each complete, correct solution to an examination question is awarded 4 course grade points. Your primary goal should be to demonstrate understanding of course topics and skill in precise mathematical formulation and solution procedures.*

1. Find bases for $C(\boldsymbol{A})$ (column space of $\boldsymbol{A}$) and $N(\boldsymbol{A})$ (null space of $\boldsymbol{A}$) with

$$\boldsymbol{A} = \begin{pmatrix} 1 & 3 & 1 & 3 & 4 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 & 2 \\ 0 & 0 & 3 & 3 & 3 \end{pmatrix}.$$

Solution. Denote $\boldsymbol{A} = (\ \boldsymbol{a}_1 \ \ \boldsymbol{a}_2 \ \ \boldsymbol{a}_3 \ \ \boldsymbol{a}_4 \ \ \boldsymbol{a}_5\ ) \in \mathbb{R}^{4\times 5} = \mathbb{R}^{m\times n}$, $m=4, n=5$. Observe that

$$\boldsymbol{a}_2 = 3\boldsymbol{a}_1, \boldsymbol{a}_4 = 2\boldsymbol{a}_1 + \boldsymbol{a}_3, \boldsymbol{a}_5 = 3\boldsymbol{a}_1 + \boldsymbol{a}_3, \tag{1}$$

and $\alpha\boldsymbol{a}_1 + \beta\boldsymbol{a}_3 = 0$ implies (2^{nd} component) $\alpha\cdot 0 + \beta\cdot 1 = 0$, hence $\beta = 0$, and then (1^{st} component) $\alpha\cdot 1 + 0\cdot 1 = 0$, hence $\alpha = 0$, and $\boldsymbol{a}_1$, $\boldsymbol{a}_3$ linearly independent. Deduce $r = \operatorname{rank}(\boldsymbol{A}) = 2$. By definition $C(\boldsymbol{A}) = \{\boldsymbol{b} \in \mathbb{R}^4 | \ \exists \boldsymbol{x} \in \mathbb{R}^5 s.t. \boldsymbol{b} = \boldsymbol{A}\boldsymbol{x}\} \le \mathbb{R}^4$, $N(\boldsymbol{A}) = \{\boldsymbol{x} \in \mathbb{R}^5 | \ \boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}\} \le \mathbb{R}^5$. By rank-nullity theorem $\dim(C(\boldsymbol{A})) = \dim(C(\boldsymbol{A}^T)) = r$, and $\dim(C(\boldsymbol{A}^T)) + \dim(N(\boldsymbol{A})) = 5 \Rightarrow \operatorname{din}(N(\boldsymbol{A})) = 3$. A basis for $C(\boldsymbol{A})$ is $\{\boldsymbol{a}_1, \boldsymbol{a}_3\}$. Rewrite (1) as

$$\begin{cases} 3\boldsymbol{a}_1 - \boldsymbol{a}_2 = \boldsymbol{0} \\ 2\boldsymbol{a}_1 + \boldsymbol{a}_3 - \boldsymbol{a}_4 = \boldsymbol{0} \\ 3\boldsymbol{a}_1 + \boldsymbol{a}_3 - \boldsymbol{a}_5 = \boldsymbol{0} \end{cases} \Rightarrow \boldsymbol{A}\boldsymbol{X} = \boldsymbol{A}\begin{pmatrix} 3 & 2 & 3 \\ -1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -5 \end{pmatrix} = \boldsymbol{0} \in \mathbb{R}^{4\times 3},$$

and columns of $\boldsymbol{X}$ are a basis for $N(\boldsymbol{A})$.

2. Define $\Pi_4 = \{\boldsymbol{a}^T\boldsymbol{b}(x) : \boldsymbol{a} \in \mathbb{R}^5, \boldsymbol{b}(x) = (\ 1 \ \ x \ \ x^2 \ \ x^3 \ \ x^4\ )^T\}$, the linear space of polynomials of degree at most 4. Determine if the following are subspaces of Π_4:

Solution. Check whether $\alpha p + \beta q \in \mathcal{S}$, $\forall\alpha, \beta \in \mathbb{R}$, $\forall p, q \in \mathcal{S}$.

a) $\mathcal{S}_a$, set of polynomials in Π_4 of even degree.

No, counterexample, $x^4+x^3, -x^4+1 \in \mathcal{S}_a$, but $(x^4+x^3)+(-x^4+1)=x^3+1 \notin \mathcal{S}_a$.

b) $\mathcal{S}_b$, set of polynomials in Π_4 of degree 3.

No, counterexample: $x^3, -x^3+1 \in \mathcal{S}_b$, $x^3+(-x^3+1)=1 \notin \mathcal{S}_b$. Also, $0 \notin \mathcal{S}_b$, hence $\mathcal{S}_b$ cannot be a subspace.

c) $\mathcal{S}_c$, set of polynomials $p \in \Pi_4$ such that $p(0)=0$.

Yes, since $\alpha(a_1x+a_2x^2+a_3x^3+a_4x^4)+\beta(b_1x+b_2x^2+b_3x^3+b_4x^4)=(\alpha a_1+\beta b_1)x+(\alpha a_2+\beta b_2)x^2+(\alpha a_3+\beta b_3)x^3+(\alpha a_4+\beta b_4)x^4 \in \mathcal{S}_c$ for any $\alpha, \beta \in \mathbb{R}$.

d) $\mathcal{S}_d$, set of polynomials in Π_4 having at least one real root.

No, counterexample: $x^2-x, x+1 \in \mathcal{S}_d$, but $x^2-x+x+1=x^2+1 \notin \mathcal{S}_d$ (imaginary roots).

3. Consider $C[-\pi,\pi]$ the vector space of continuous functions defined on $[-\pi,\pi]$, the inner product

$$\langle f, g\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, g(x)\,\mathrm{d}x, \tag{2}$$

and the induced norm $\|f\| = \langle f, f\rangle^{1/2}$, $f, g \in C[-\pi,\pi]$.

a) Are $\cos x$ and $\sin x$ orthogonal unit vectors with respect to the inner product (2)?

True if $\|\cos x\| = \|\sin x\| = 1$, and $\langle \cos x, \sin x\rangle = 0$. Compute

$$\|\cos x\|^2 = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos^2 x\,\mathrm{d}x = \frac{1}{2\pi}\int_{-\pi}^{\pi} (1+\cos 2x)\,\mathrm{d}x = 1,$$

$$\|\sin x\|^2 = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos^2 x\,\mathrm{d}x = \frac{1}{2\pi}\int_{-\pi}^{\pi} (1-\cos 2x)\,\mathrm{d}x = 1,$$

$$\langle \cos x, \sin x\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos x \sin x\,\mathrm{d}x = \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin 2x\,\mathrm{d}x = 0.$$

Yes, properties are verified.

b) What are the angles between x and $\cos x$, $\sin x$?

By definition the angles are

$$\theta = \arccos\frac{\langle x, \cos x\rangle}{\|x\|\,\|\cos x\|}, \quad \varphi = \arccos\frac{\langle x, \sin x\rangle}{\|x\|\,\|\sin x\|}.$$

Compute

$$\|x\| = \left(\frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,\mathrm{d}x\right)^{1/2} = \left(\frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,\mathrm{d}x\right)^{1/2} = \pi\left(\frac{2}{3}\right)^{1/2}$$

$$\langle x, \cos x\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos x\,\mathrm{d}x = 0, \langle x, \sin x\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos x\,\mathrm{d}x = 2,$$

hence $\theta = \pi/2$, and $\varphi = \arccos\left(\sqrt{6}/\pi\right)$.