Homework 1

Due date: Feb 3, 2017, 11:55PM.

Bibliography: Course lecture notes Lessons 1-8.

Note: Save this file as `Homework1Solution.tm` in your `/home/student/courses/MATH547/homework` directory. Then carry out your work on solving the problems in TeXmacs. Submit your final version through Sakai. If you've understood the theoretical concepts reviewed in the Background section, each problem should require about 20 minutes to complete, hence the entire homework can be completed within 3 hours. The exercises are from p.131 of the textbook, an image of which is included at the end.

1 Background

1.1 Objectives

In Homework 1 you start exploration of the the mathematical constructs introduced to answer the question: "What are the solutions of the linear system $\boldsymbol{A}\boldsymbol{x}=\boldsymbol{b}$?". The goal of this homework is to understand the concepts of vector spaces, in particular the fundamental spaces associated with a matrix and their significance. The four theory problems investigate these concepts both through symbolic exercises and small numerical applications. Small examples of how to solve the requested exercises are included. The computer application mirrors these questions using musical notation as one example of the many uses of linear algebra. These concepts build up to the *Rank Nullity Theorem*, or *Fundamental Theorem of Linear Algebra* that answers the question on existence and number of solutions to a linear system.

1.2 Concept review

1.2.1 Column, null spaces

An unfortunate consequence of the wide-spread use of linear algebra is that multiple names are used for the same concept. In particular, for a matrix $\boldsymbol{A}\in\mathbb{R}^{m\times n}$, $\boldsymbol{A}=(\boldsymbol{a}_1 \ldots \boldsymbol{a}_n)$, the *image* of $\boldsymbol{A}$ is the same as the *column space* or *range* or *span of columns* of $\boldsymbol{A}$

$$\operatorname{im}(\boldsymbol{A})=C(\boldsymbol{A})=\operatorname{range}(\boldsymbol{A})=\operatorname{span}\{\boldsymbol{a}_1,\ldots,\boldsymbol{a}_n\}=\{\boldsymbol{b}\in\mathbb{R}^m \mid \exists\boldsymbol{x}\in\mathbb{R}^n \text{ such that } \boldsymbol{b}=\boldsymbol{A}\boldsymbol{x}\}\leq\mathbb{R}^m.$$

Similarly the *kernel* of $\boldsymbol{A}$ is the same as the *null space* of $\boldsymbol{A}$

$$N(\boldsymbol{A})=\operatorname{null}(\boldsymbol{A})=\{\boldsymbol{x}\in\mathbb{R}^n \mid \boldsymbol{A}\boldsymbol{x}=\boldsymbol{0}\}\leq\mathbb{R}^n.$$

We shall use $C(\boldsymbol{A}), N(\boldsymbol{A})$ as the preferred notation.

1.2.2 Vector spaces, subspaces

Also, *vector spaces* and *linear spaces* refer to the same mathematical concept, often denoted as $(\mathcal{V},\mathcal{S},+)$ consisting of a set of vectors $\mathcal{V}$, a set of scalars $\mathcal{S}$, an addition operation between vectors and a multiplication operation between scalars and vectors that satisfy the following properties for arbitrary $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}\in\mathcal{V}$, $\alpha,\beta\in\mathcal{S}$

Closed. $\boldsymbol{u}+\boldsymbol{v}\in\mathcal{V}$

Associativity. $\boldsymbol{u}+(\boldsymbol{v}+\boldsymbol{w})=(\boldsymbol{u}+\boldsymbol{v})+\boldsymbol{w}$

Null element. $\exists\boldsymbol{0}\in\mathcal{V}$ such that $\boldsymbol{u}+\boldsymbol{0}=\boldsymbol{u}$

Inverse element. $\exists(-\boldsymbol{u})$ such that $\boldsymbol{u}+(-\boldsymbol{u})=\boldsymbol{0}$

Commutativity. $\boldsymbol{u}+\boldsymbol{v}=\boldsymbol{v}+\boldsymbol{u}$

Distributivity over scalar addition. $(\alpha+\beta)\boldsymbol{u}=\alpha\boldsymbol{u}+\beta\boldsymbol{v}$

Distributivity over vector addition. $\alpha(\boldsymbol{u}+\boldsymbol{v})=\alpha\boldsymbol{u}+\alpha\boldsymbol{v}$

Scalar identity. $1\in\mathcal{S}\Rightarrow 1\boldsymbol{u}=\boldsymbol{u}$

It is very useful to separate a vector space into "smaller" parts. The relevant mathematical construct is that of a vector subspace

Definition. *$\mathcal{U}$ is a vector subspace of vector space $\mathcal{V}$ over the same scalar field $\mathcal{S}$ if for any $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{U}$, any $\alpha, \beta \in \mathcal{S}$*

Inclusion. *$\boldsymbol{u} \in \mathcal{V}$ (a vector in the subspace is also in the enclosing vector space)*

Closed. *$\alpha \boldsymbol{u} + \beta \boldsymbol{v} \in \mathcal{U}$ (linear combinations of subspace vectors stay within the subspace)*

1.2.3 Linear transformations, mapping

Matrices are closely related to the concept of a linear transformation or linear mapping

Definition. *A linear mapping is a function between two vector spaces $\mathcal{X}, \mathcal{Y}$ over the same scalar field $\mathcal{S}$, f: $\mathcal{X} \mapsto \mathcal{Y}$, with the property that $\forall \boldsymbol{u}, \boldsymbol{v} \in \mathcal{X}$, $\forall \alpha, \beta \in \mathcal{S}$*

$$f(\alpha \boldsymbol{u} + \beta \boldsymbol{v}) = \alpha f(\boldsymbol{u}) + \beta f(\boldsymbol{v})$$

A matrix $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ is a linear mapping from the vector space $(\mathbb{R}^n, \mathbb{R}, +)$ to the vector space $(\mathbb{R}^m, \mathbb{R}, +)$

$$\boldsymbol{A} \colon \mathbb{R}^n \mapsto \mathbb{R}^m$$

$$\boldsymbol{x} \in R^n, \boldsymbol{b} \in \mathbb{R}^m, \boldsymbol{b} = \boldsymbol{A}\boldsymbol{x}$$

The fundamental spaces $C(\boldsymbol{A}), N(\boldsymbol{A})$ are vector subspaces of $\mathbb{R}^m, \mathbb{R}^n$ respectively, which we denote by

$$C(\boldsymbol{A}) \leq \mathbb{R}^m, N(\boldsymbol{A}) \leq \mathbb{R}^n.$$

1.2.4 Linear independence, basis set

The basic operation within linear algebra is the linear combination $\boldsymbol{A}\boldsymbol{x}$, and one fundamental question is what vectors $\boldsymbol{b}$ are reachable by a linear combination (i.e., when does $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ have a solution). In addition to the previous concepts, the final mathematical tool used to answer this question is that of linear independence

Definition. *The vectors $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n \in \mathcal{V}$, are linearly independent if the only n scalars, $x_1, \ldots, x_n \in \mathcal{S}$, that satisfy*

$$x_1 \boldsymbol{a}_1 + \ldots x_n \boldsymbol{a}_n = \boldsymbol{0}, \tag{1}$$

are $x_1 = 0$, $x_2 = 0, \ldots, x_n = 0$.

Notice that linear independence essentially states a relationship between the vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$, namely that none of these vectors can be reached by a linear combination of the others.

Definition. *A set of vectors $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n \in \mathcal{V}$ is a basis for vector space $\mathcal{V}$ if:*

1. *$\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n$ are linearly independent;*
2. *$\operatorname{span}\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n\} = \mathcal{V}$.*

2 Theory problems

2.1 Vector spaces

Solve exercises 3.2.2-6. from the textbook, p.131. Solution to exercise 3.2.1 is presented as an example. Complete the rest.

Ex3.2.1 Solution. Verify subspace property $\forall \boldsymbol{u}_1, \boldsymbol{u}_2 \in W, \forall \alpha_1, \alpha_2 \in \mathbb{R} \Rightarrow \alpha_1 \boldsymbol{u}_1 + \alpha_2 \boldsymbol{u}_2 \in \mathbb{R}$. Since $\boldsymbol{u}_1, \boldsymbol{u}_2 \in W$, we have

$$\boldsymbol{u}_1 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}, \boldsymbol{u}_2 = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}, \text{with } x_1 + y_1 + z_1 = 1 \text{ and } x_2 + y_2 + z_2 = 1.$$

Compute

$$\boldsymbol{u} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \boldsymbol{u}_1 + \boldsymbol{u}_2 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{pmatrix}.$$

Note that $x+y+z=(x_1+x_2)+(y_1+y_2)+(z_1+z_2)=(x_1+y_1+z_1)+(x_2+y_2+z_2)=1+1=2$, hence $\boldsymbol{u}\notin W$, and W is not a subspace since it does not satisfy closure property.

Ex3.2.2 Solution.

Ex3.2.3 Solution.

Ex3.2.4 Solution.

Ex3.2.5 Solution.

Ex3.2.6 Solution.

2.2 Linear independence

Solve exercises 3.2.14-3.2.18 *either* through hand computation (until you're confident of the procedure) or using Octave. Solutions to 3.2.19-20 are presented as examples

Ex3.2.19 Solution. Set

$$\boldsymbol{A}=\begin{pmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \boldsymbol{a}_3 & \boldsymbol{a}_4 & \boldsymbol{a}_5 \end{pmatrix}=\begin{pmatrix} 1 & 2 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Note $\boldsymbol{a}_2=2\boldsymbol{a}_1$, $\boldsymbol{a}_5=3\boldsymbol{a}_1+4\boldsymbol{a}_3+5\boldsymbol{a}_4$, hence vectors $\{\boldsymbol{a}_1,\boldsymbol{a}_2,\boldsymbol{a}_3,\boldsymbol{a}_4,\boldsymbol{a}_5\}$ are linearly dependent. Vectors $\boldsymbol{a}_1,\boldsymbol{a}_3,\boldsymbol{a}_4$ are linearly independent and $\boldsymbol{a}_2,\boldsymbol{a}_5$ are redundant. Using Octave, the number of linearly independent column vectors is the rank of the matrix

```
octave> A=[1 2 0 0 3; 0 0 1 0 4; 0 0 0 1 5; 0 0 0 0 0]
   A =

      1   2   0   0   3
      0   0   1   0   4
      0   0   0   1   5
      0   0   0   0   0

octave> rank(A)
   ans =  3
octave> rank([A(:,1) A(:,3) A(:,4)])
   ans =  3
octave>
```

Ex3.2.20 Solution. Place the vectors as rows and reduce to row echelon form by hand computation

$$\boldsymbol{A}^T=\begin{pmatrix} \boldsymbol{a}_1^T & \boldsymbol{a}_2^T & \boldsymbol{a}_3^T \end{pmatrix}=\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 4 & 7 & 10 \end{pmatrix}\sim\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 3 & 6 & 9 \end{pmatrix}\sim\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}\sim\begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Deduce that $\boldsymbol{a}_1,\boldsymbol{a}_2$ are linearly independent, but $\boldsymbol{a}_1,\boldsymbol{a}_2,\boldsymbol{a}_3$ are linearly dependent. In Octave

```
octave> A=[1 1 1; 1 2 4; 1 3 7; 1 4 10]
```

```
A =

   1   1   1
   1   2   4
   1   3   7
   1   4  10

octave> rank(A')
ans =  2
octave> rref(A')
ans =

   1   0  -1  -2
   0   1   2   3
   0   0   0   0

octave>
```

Find coefficients that express $\boldsymbol{a}_3$ as a linear combination of $\boldsymbol{a}_1, \boldsymbol{a}_2$

```
octave> x=[A(:,1) A(:,2)] \ A(:,3)
x =

  -2
   3

octave> [x(1)*A(:,1)+x(2)*A(:,2) A(:,3)]
ans =

    1    1
    4    4
    7    7
   10   10

octave>
```

Ex3.2.14 Solution.

Ex3.2.15 Solution.

Ex3.2.16 Solution.

Ex3.2.17 Solution.

Ex3.2.18 Solution.

2.3 Null space

Solve exercises 3.2.22-3.2.26 *either* through hand computation (until you're confident of the procedure) or using Octave. Solution to 3.2.21 is presented as an example

Ex3.2.21 Solution. Denote

$$\boldsymbol{A} = (\ \boldsymbol{a}_1\ \ \boldsymbol{a}_2\ ) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$

and notice $\boldsymbol{a}_1 = \boldsymbol{a}_2$, hence $\boldsymbol{a}_1 - \boldsymbol{a}_2 = \boldsymbol{0}$, or

$$\boldsymbol{A}\boldsymbol{x} = 0, \text{with } \boldsymbol{x} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

and $\boldsymbol{x} \in N(\boldsymbol{A})$. In Octave the `null` function provides a basis for the null space

```
octave> A=[1 1; 1 1]
   A =

      1   1
      1   1

octave> null(A)
   ans =

     -0.70711
      0.70711

octave>
```

The `null` function always returns an orthonormal set of vectors.

Ex3.2.22 Solution.

Ex3.2.23 Solution.

Ex3.2.24 Solution.

Ex3.2.25 Solution.

Ex3.2.26 Solution.

2.4 Column space

Solve exercises 3.2.29-3.2.33 *either* through hand computation (until you're confident of the procedure) or using Octave. Solutions to 3.2.27-28 are presented as examples.

Ex3.2.27 Solution. Denote

$$\boldsymbol{A} = (\ \boldsymbol{a}_1\ \ \boldsymbol{a}_2\ ) = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix}$$

and carry out reduction to row echelon form of $\boldsymbol{A}^T$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

to deduce that $\text{rank}(\boldsymbol{A}) = 2$ (two non-zero pivots), hence $\boldsymbol{a}_1, \boldsymbol{a}_2$ are independent and form a basis for $C(\boldsymbol{A})$. In Octave the `orth` function provides an orthonormal basis set for $C(\boldsymbol{A})$.

```
octave> A=[1 1; 1 2; 1 3]
  A =

     1   1
     1   2
     1   3

octave> B=orth(A)
  B =

     0.32311  -0.85378
     0.54751  -0.18322
     0.77190   0.48734

octave>
```

To express the basis set found by hand computation in terms of $\boldsymbol{b}_1, \boldsymbol{b}_2$, solve the linear systems

```
octave> x=B \ A(:,1)
  x =

     1.64252
    -0.54966

octave> x(1)*B(:,1)+x(2)*B(:,2)
  ans =

     1
     1
     1

octave> y=B \ A(:,2)
  y =

     3.73384
     0.24180

octave> y(1)*B(:,1)+y(2)*B(:,2)
  ans =

     1.0000
     2.0000
     3.0000

octave>
```

Ex3.2.28 Solution.

Ex3.2.29 Solution.

Ex3.2.30 Solution.

Ex3.2.31 Solution.

Ex3.2.32 Solution.

Ex3.2.33 Solution.

3 Computational problems

We now seek to reinforce the theoretical concepts and small computations by a more realistic application using music notation. For this part use the score associated with your section and this linked table of frequencies. The intent of this part of the homework is to invite you to think carefully about how theoretical concepts map to a particular application.

In class we use *Mathematica* to demonstrate construction of vectors of sound pressure since it allows for playback and has predefined note frequencies. Octave does not have all these features, but it is simple to construct the notes within the first measure of *Ode to Joy*.

Define the frequencies for the first four notes E4,F4,G4

```
octave> freqE4=329.63; freqF4=349.23; freqG4=392.00;
octave>
```

Define a sampling rate, a time duration for a measure, and a vector of time values that corresponds to a full measure

```
octave> s=0.0001; T=0.1; t=(0:s:T-s)'; m=size(t)
  m =

     1000        1

octave>
```

In the above we use the Octave language construct `start:incr:end`, as exemplified by

```
octave> 2:3:15
  ans =

      2    5    8   11   14

octave>
```

Define windows that correspond to each beat within a measure.

```
octave> beat=zeros([m,4]);
octave> beat(1:m/4,1)=1; beat(m/4+1:m/2,2)=1; beat(m/2+1:3*m/4,3)=1; beat(3*m/4+1:m,4)=1;
octave>
```

Define vectors that play E4 in beat 1, E4 in beat 2, F4 in beat 3, G4 in beat 4. We use the notation nNNbB with NN the note and B the beat

```
octave> nE4b1 = beat(:,1).*sin(2*pi*freqE4*t);
octave> nE4b2 = beat(:,2).*sin(2*pi*freqE4*t);
octave> nF4b3 = beat(:,3).*sin(2*pi*freqF4*t);
octave> nG4b4 = beat(:,4).*sin(2*pi*freqG4*t);
octave>
```

The first measure in Ode to Joy contains a forte symbol. Assume that corresponds to playing the note at 50% more volume

```
octave> forte=1.5; piano=0.5;
octave>
```

The first measure in Ode to Joy is a linear combination

```
octave> OtJm01 = forte*nE4b1 + nE4b2 + nF4b3 + nG4b4;
octave>
```

Additional notes can be defined by the same procedure. For the second measure of Ode to Joy we'd need

```
octave> nG4b1 = beat(:,1).*sin(2*pi*freqG4*t);
octave> nF4b2 = beat(:,2).*sin(2*pi*freqF4*t);
octave> nE4b3 = beat(:,3).*sin(2*pi*freqE4*t);
octave> freqD4=293.66; nD4b4 = beat(:,4).*sin(2*pi*freqD4*t);
octave> OtJm02 = nG4b1 + nF4b2 + nE4b3 + nD4b4;
octave>
```

We now ask the exact same questions as in the theory part, but with respect to musical notation. For your homework, use the score associated with your class section.

3.1 Vector spaces

Is measure n within the span of measures $1, 2, \ldots n-1$? Here's a very straightforward example answer for $n=2$.

```
octave> rank([OtJm01 OtJm02])
  ans =  2
octave>
```

Since the rank is 2 the two vectors are linearly independent, and the second measure is not in the span of measure 1. Provide answers for $n=3, 4, 5$.

3.2 Linear independence

Are measures (i, j) linearly independent for $1 \leqslant i, j \leqslant 4$?

3.3 Null space

Construct a matrix $\boldsymbol{A}$ containing measures 1-6. What is $N(\boldsymbol{A})$? Think first about the problem, and present an answer based on your analysis. Then carry out a computation to confirm.

3.4 Column space

Construct a matrix $\boldsymbol{A}$ containing measures 1-6. What is $C(\boldsymbol{A})$? Think first about the problem, and present an answer based on your analysis. Then carry out a computation to confirm.