Homework 5

Due date: April 5, 2017, 11:55PM.

Bibliography: Course lecture notes Lessons 19-23.

Note: Save this file as `Homework5Solution.tm` in your `/home/student/courses/MATH547/homework` directory. Then carry out your work on solving the problems in TeXmacs. Submit your final version through Sakai. If you've understood the theoretical concepts reviewed in the Background section and lecture notes, each problem should require about 20 minutes to complete, hence the entire homework can be completed within 3 hours.

1 Background

This assignment concentrates on solving the eigenvalue problem $\boldsymbol{A}\boldsymbol{x}=\lambda\boldsymbol{x}$, with $\boldsymbol{A}\in\mathbb{R}^{m\times m}$, $\lambda\in\mathbb{C}$, $\boldsymbol{x}\in\mathbb{C}^m$.

2 Theory problems

2.1 Eigenvalues, eigenvectors of related matrices

Often knowledge of the solution of $\boldsymbol{A}\boldsymbol{x}=\lambda\boldsymbol{x}$ allows easy solution of some related eigenproblems. Solve 7.1.1-7.1.4

Ex7.1.1 Solution.

Ex7.1.2 Solution.

Ex7.1.3 Solution.

Ex7.1.4 Solution.

2.2 Geometric significance of eigenbases

Sometimes, the action of matrices upon vectors has straightforward geometric interpretations. In this case an eigenbasis for the matrix can be determined from geometric considerations.

Ex7.1.58 Solution.

Ex7.1.59 Solution.

Ex7.1.60 Solution.

Ex7.1.62 Solution.

2.3 Eigenvalue computation

Determine the eigenvalues for these problems by hand computation. Show your work. Verify your result in Octave.

Ex7.2.9 Solution.

Ex7.2.10 Solution.

Ex7.2.11 Solution.

Ex7.2.12 Solution.

2.4 Eigenvector computation

Determine the eigenvalues eigenvectors by hand computation. Show your work. Use insight into the structure of the problem.

Ex7.3.17 Solution.

Ex7.3.18 Solution.

Ex7.3.19 Solution.

Ex7.3.20 Solution.

3 Computational problems

This assignment and the next illustrate the applications of eigenvalues and the singular value decomposition in the face recognition problem. The following instructions load a matrix $\boldsymbol{A} \in \mathbb{R}^{m \times n}$, containing $n=99$ columns, where adding the average $\boldsymbol{a} \in \mathbb{R}^m$ to columns of $\boldsymbol{A}$ gives an image with $m=2^{14}=16384$ pixels. Also loaded are matrices $\boldsymbol{R}, \boldsymbol{S} \in \mathbb{R}^{n \times p}$ of $n=99$ coefficients of linear combinations of columns of $\boldsymbol{A}$ to give $p=2000$ face images.

```
octave> cd /home/student/courses/MATH547/homework/
octave> load ../lessons/mitfaces/faces.mat
octave> who
   Variables in the current scope:

   A        R        S        a        ans      dispans  prompt   r

octave> [size(A) size(a) size(R) size(S)]
   ans =

      16384      99    16384       1      99     2000      99     2000

octave> [m,n]=size(A); [n,p]=size(R);
octave> [m n p]
   ans =

      16384      99     2000

octave>
```

A vector of length m, say the average face $\boldsymbol{a}$, is displayed through the instructions

```
octave> imagesc(flip(reshape(a,128,128)',1)); colormap(gray(256)); print -dpng aface.png
octave>
```

An image constructed from a linear combination of columns of $\boldsymbol{A}$, say with coefficients $\boldsymbol{r}_1$ the first column of $\boldsymbol{R}$ is computed and displayed through

```
octave> f = A*R(:,1)+a;
```

```
octave> imagesc(flip(reshape(f,128,128)',1)); colormap(gray(256)); print -dpng r1face.png
octave>
```

An image, e.g. image 1223, from which the database was constructed can be visualized through

```
octave> fid=fopen('/home/student/courses/MATH547/lessons/mitfaces/rawdata/1223');
octave> face=fread(fid); fclose(fid);
octave> imagesc(flip(reshape(face,128,128)',1)); colormap(gray(256)); print -dpng
        1223face.png
octave>
```

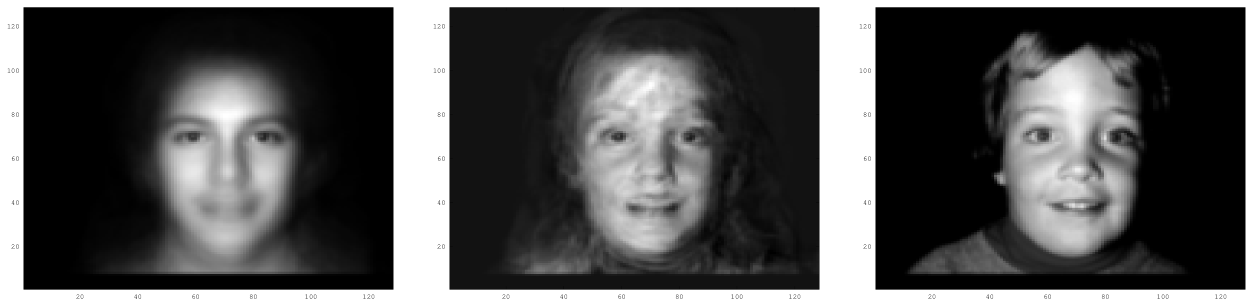

Figure 1. Left: average face among database subjects. Middle: face constructed from linear combination of database subjects. Right

Muliple images, e.g. from 1300 to 1350, can be read through the following instructions

```
octave> basedir = '/home/student/courses/MATH547/lessons/mitfaces/rawdata/';
octave> n1=1301; n2=1350; nim=n2-n1+1; m=16384; B=zeros([m,nim]);
octave> for i=n1:n2
          fid = fopen(strcat(basedir,num2str(i)));
          B(1:m,i-n1+1) = fread(fid); fclose(fid);
        end
octave> size(B)
   ans =

      16384       50

octave>
```

The matrix $\boldsymbol{A}$ from the data is constructed from a certain subset of the available faces. This assignment focuses on constructing a different matrix $\boldsymbol{B} \in \mathbb{R}^{m \times q}$ from a different subset, and comparing the two representations.

1. Choose a subset of `q=50` files from the available faces (e.g., choose some `n1, n2=49+n1`, do not choose `n1=1301`). Use the script above to the read the faces into a matrix $\boldsymbol{B} \in \mathbb{R}^{m \times q}$. Compute a vector $\boldsymbol{b} \in \mathbb{R}^m$ that is the average of the chosen vectors. Use the `sum` Octave function. Display the average image.
2. The correlation matrix is defined as $\boldsymbol{C} = \frac{1}{q}\boldsymbol{M}\boldsymbol{M}^T$, with column $\boldsymbol{m}_i$ of $\boldsymbol{M} \in \mathbb{R}^{m \times q}$ defined in terms of column $\boldsymbol{b}_i$ of $\boldsymbol{B}$ and the average as $\boldsymbol{m}_i = \boldsymbol{b}_i - \boldsymbol{b}$, for $i = 1, \ldots, q$. What is the size of $\boldsymbol{C}$? Find the relationship between eigenvalues, eigenvectors of $\boldsymbol{C}$ and those of $\boldsymbol{D} = \frac{1}{q}\boldsymbol{M}^T\boldsymbol{M}$. Which is easier to compute, the eigenvalues of $\boldsymbol{C}$ or those of $\boldsymbol{D}$?
3. Compute the eigenvalues and eigenvectors of $\boldsymbol{D}$. Use these and your result from 2 to compute the eigenvalues and eigenvectors of $\boldsymbol{B}$. Place the normalized eigenvectors in a matrix $\boldsymbol{X} \in \mathbb{R}^{m \times q}$. Are the columns of $\boldsymbol{X}$ orthogonal to one another?

4. Choose one of the faces within the dataset, i.e., choose j, $\boldsymbol{f} = \boldsymbol{b}_j - \boldsymbol{b}$. Find the projection of this vector onto $C(\boldsymbol{X})$. Denote the projection as $\boldsymbol{X}\boldsymbol{u}$. It solves the least squares problem $\min_{\boldsymbol{u}} \|\boldsymbol{f} - \boldsymbol{X}\boldsymbol{u}\|$. Display the images obtained as $\boldsymbol{v} = \sum_{j=1}^{r} u_j \boldsymbol{x}_j$ for $r = 2, 4, 8, 16, 32$. Comment on the result.

Note: As usual, the above questions can be answered using very few lines of Octave code, no more that 5 lines per question. The main effort is properly conceptualizing the linear algebra operations, and thinking in terms of vectors and matrix operations. Pay particular attention to the dimension of the result from every matrix multiplication.