Homework 3

Due date: Feb 24, 2017, 11:55PM.

Bibliography: Course lecture notes Lessons 11-15. Textbook section 5.5

Note: Save this file as `Homework3Solution.tm` in your `/home/student/courses/MATH547/homework` directory. Then carry out your work on solving the problems in TeXmacs. Submit your final version through Sakai. If you've understood the theoretical concepts reviewed in the Background section and lecture notes, each problem should require about 30 minutes to complete, hence the entire homework can be completed within 4 hours.

1 Background

1.1 Objectives, basic concepts

Up to this point in the course, linear algebra concepts have been presented in the context of vectors within $\mathbb{R}^m$. This assignment highlights the general applicability of linear algebra concepts through problems in inner product spaces. The inner (or scalar) product is a generalization of the dot product between vectors

2 Theory problems

2.1 Inner product examples

Solve exercises 5.5.3, 4, 14, 19, from textbook, p.260-261. Solution to exercise 5.5.15. is presented as an example. Complete the rest.

Ex5.5.15 Solution. Check whether inner product properties are satisfied for $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathbb{R}^2$:

– symmetry: $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = x_1 y_1 + b x_1 y_2 + c x_2 y_1 + d x_2 y_2$,$\langle \boldsymbol{y}, \boldsymbol{x} \rangle = y_1 x_1 + b y_1 x_2 + c y_2 x_1 + d y_2 x_2$

$$D = \langle \boldsymbol{x}, \boldsymbol{y} \rangle - \langle \boldsymbol{y}, \boldsymbol{x} \rangle = b(x_1 y_2 - y_1 x_2) + c(x_2 y_1 - y_2 x_1) = (b-c) x_1 y_2 - (b-c)(y_1 x_2)$$

$D = 0$ for $b = c$. Assume $c = b$ henceforth.

– distributivity with respect to vector addition:

$$\begin{array}{l} D = \langle \boldsymbol{x} + \boldsymbol{y}, \boldsymbol{z} \rangle - \langle \boldsymbol{x}, \boldsymbol{z} \rangle - \langle \boldsymbol{y}, \boldsymbol{z} \rangle, \\ \langle \boldsymbol{x} + \boldsymbol{y}, \boldsymbol{z} \rangle = (x_1 + y_1) z_1 + b(x_1 + y_1) z_2 + b(x_2 + y_2) z_1 + d(x_2 + y_2) z_2, \\ \langle \boldsymbol{x}, \boldsymbol{z} \rangle = x_1 z_1 + b x_1 z_2 + b x_2 z_1 + d x_2 z_2, \\ \langle \boldsymbol{y}, \boldsymbol{z} \rangle = y_1 z_1 + b y_1 z_2 + b y_2 z_1 + d x_2 y_2, \end{array} \Rightarrow D = 0 \checkmark$$

Cancellation of terms shown through color coding.

– distributivity with respect to scalar multiplication:

$$D = a \langle \boldsymbol{x}, \boldsymbol{y} \rangle - \langle a \boldsymbol{x}, \boldsymbol{y} \rangle = a(x_1 y_1 + b x_1 y_2 + b x_2 y_1 + d x_2 y_2) - (a x_1 y_1 + b a x_1 y_2 + b a x_2 y_1 + d a x_2 y_2) = 0. \checkmark$$

– positive definiteness: $\langle \boldsymbol{x}, \boldsymbol{x} \rangle = x_1^2 + 2bx_1x_2 + dx_2^2 = (x_1 + bx_2)^2 + (d - b^2)x_2^2$. Since $\langle \boldsymbol{x}, \boldsymbol{x} \rangle$ should be positive for *any* $\boldsymbol{x} \neq \mathbf{0}$, deduce that $d > b^2$, in which case $\langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0$ would imply $x_2 = 0$, and subsequently $x_1 = 0$.

Ex5.5.3 Solution.

Ex5.5.4 Solution.

Ex5.5.19 Solution.

Ex5.5.14 Solution.

2.2 Orthonormalization in inner product spaces

Solve exercises 5.5.10, 16

Ex5.5.10 Solution.

Ex5.5.16 Solution.

2.3 Standard operations in inner product spaces

Solve 5.5.24

Ex5.5.24 Solution.

2.4 Fourier coefficients of the Heaviside function

Solve exercise 5.5.27. The function in this exercise is known as the Heaviside function. The fact that a a jump can be represented as a sum of smooth trigonometric functions generated enormous debate during the 19th century during the development of Fourier analysis

Ex5.5.27 Solution.

3 Computational problems

We reinforce theoretical concepts by again appealing to music. In Homework 1 you considered the representation of single notes in a musical score as vectors of pressure values. In this homework, we start from a short recording and represent the signal $f(t)$ as a linear combination of trigonometric functions (Fourier analysis).

Choose a short (1-5 sec) piece of music. Samples will be provided and discussed in class. Assume the music repeats after time T, hence $f(t)=f(t+T)$ (f is periodic with period T)

3.1 Inner product

Compute the inner products

$$a_k=\frac{2}{T}\int_0^T f(t)\cos\left(\frac{2\pi kt}{T}\right)\mathrm{dt},\ b_k=\frac{2}{T}\int_0^T f(t)\sin\left(\frac{2\pi kt}{T}\right)\mathrm{dt},\ k=1,\ldots,1000$$

Recognize that the above are particular cases of the scalar product

$$\langle f,g\rangle=\frac{2}{T}\int_0^T f(t)\,g(t)\,\mathrm{dt}.$$

Upon sampling f,g, obtain the values $f_j=f(t_j)$, $g=g(t_j)$, $j=1,2,\ldots,m$, $t_j=jh$, $h=T/m$. The above scalar product can then be approximated by the (Darboux) sum

$$\langle f,g\rangle\cong\frac{2h}{T}\sum_{j=1}^{m} f_j g_j=\frac{2}{m}\sum_{j=1}^{m} f_j g_j.$$

Introducing vectors for the samples $\boldsymbol{f},\boldsymbol{g}$, obtain

$$\langle f,g\rangle\cong\frac{2}{m}\boldsymbol{f}^T\boldsymbol{g},$$

the familiar dot product, and a good approximation of the inner product can be readily obtained in Octave, exemplified below for the computation of a_1,b

```
octave> m=20000; T=3.; h=T/m; t=(1:m)'*h; f=2*cos(2*pi*t/T)+4*sin(2*pi*2*t/T);
octave> size(f)
   ans =

      20000        1

octave> a1=2/m*f'*cos(2*pi*t/T)
   a1 =  2.0000
octave> b2=2/m*f'*sin(2*pi*2*t/T)
   b2 =  4.0000
octave>
```

Write a loop to compute vectors $\boldsymbol{a},\boldsymbol{b}\in\mathbb{R}^n$, $n=1000$. Plot the resulting vector components, (k,a_k), (k,b_k), and the norm $\left(k,\sqrt{a_k^2+b_k^2}\right)$.

Solution. Following procedures presented in class, load the mat file containing the Chopin Waltz in A minor. Recall that the sound is sampled at $f = 44.1\,\text{kHz}$, so if we have n samples the length of the piece is $T = N/f$ seconds. Construct a vector y containing a clip 0.25 seconds long, starting halfway through the piece, from t_0 to t_1. Construct a vector of the sample times t, and plot the clip.

```
octave> cd /home/student/courses/MATH547
octave> load lessons/musicfiles/ChopinWaltzAminor
octave> who
   Variables in the current scope:

   Expression1  ans         dispans     prompt      r

octave> N=max(size(Expression1))
   N =  4862592
octave> f=44100; i0=N/2+1; i1=floor(N/2+0.25*f); y=Expression1(i0:i1); m=max(size(y))
   m =  11025
octave> T=N/f; t0=i0/f; t1=i1/f; [T t0 t1]
   ans =

      110.263    55.131    55.381

octave> t=(t0:1/f:t1)'-t0; plot(t,y); xlabel('t'); ylabel('y');
octave> print -dpng homework/HW3Fig1.png
octave>
```

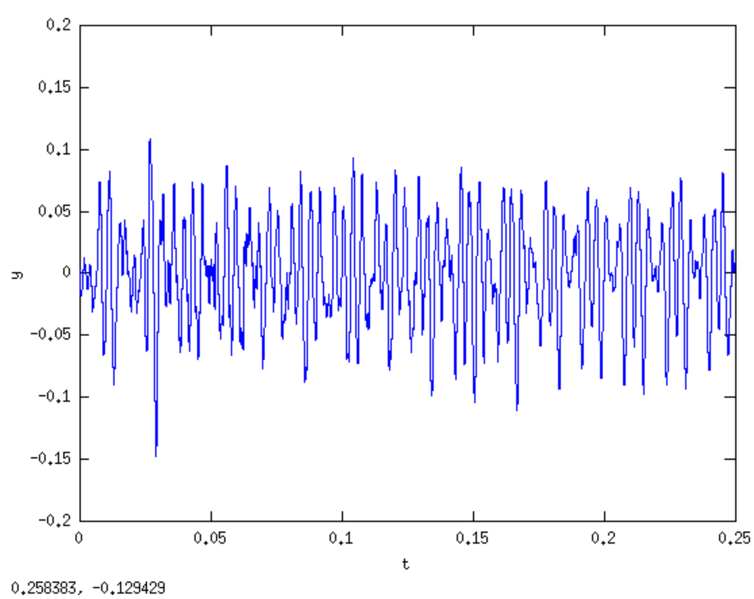

Figure 1. Sound clip from Chopin waltz in A minor. The plot displayed by Octave is captured using the Screenshot utility.

Operations up to this point just read the data, and were covered in class. We turn now to a concise solution of what is asked in this problem.

Construct matrices $\boldsymbol{C}, \boldsymbol{S} \in \mathbb{R}^{m \times n}$ containing the trigonometric basis functions

$$\boldsymbol{C} = \left(\begin{array}{cccc} \cos(t) & \cos(2t) & \ldots & \cos(nt) \end{array}\right), \boldsymbol{S} = \left(\begin{array}{cccc} \sin(t) & \sin(2t) & \ldots & \sin(nt) \end{array}\right)$$

with $t \in \mathbb{R}^m$ the column vector of sample times, and compute the requested coefficients

$$\boldsymbol{a} = \frac{2}{m} \boldsymbol{y}^T \boldsymbol{C}, \boldsymbol{b} = \frac{2}{m} \boldsymbol{y}^T \boldsymbol{S},$$

and compute $c_k = \sqrt{a_k^2 + b_k^2}$.

The Octave operations just require understanding of matrix multiplication. Four lines solves the problem

```
octave> n=1000; C=cos(2*pi*t*(1:n)); S=sin(2*pi*t*(1:n));
octave> a=(2/m)*y'*C; b=(2/m)*y'*S; c=sqrt(a.^2+b.^2);
octave> plot(1:n,a,'.r',1:n,b,'.b',1:n,c,'.k'); xlabel('k'); ylabel('ak,bk,ck');
octave> print -dpng HW3Fig2.png
octave>
```

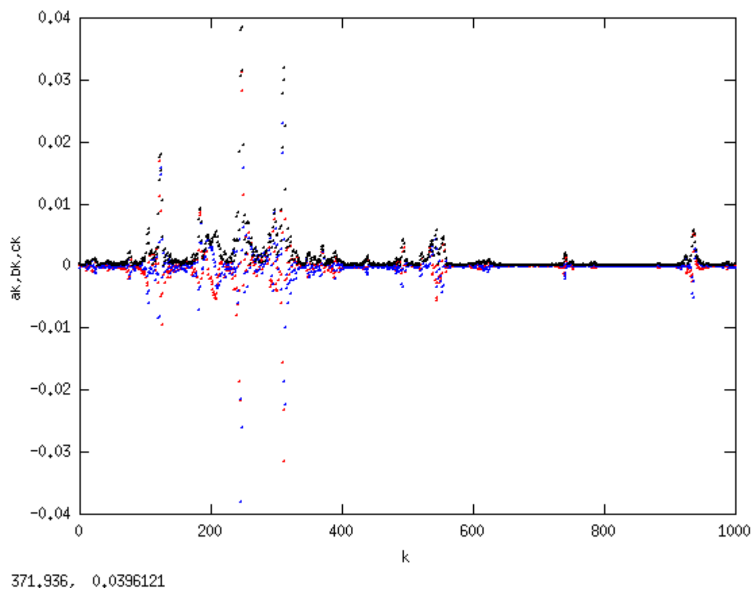

Figure 2. Fourier coefficients. The dominant notes seem to be at $n \cong 440$ Hz and $n \cong 820$ Hz

3.2 Filtering

A very common operation is filtering, or elimination of some of the Fourier components of a signal. In linear algebra terms, this is an operation we've already encountered: it's a projection on a subset of the Fourier basis set $\{1, \cos(2\pi t/T), \sin(2\pi t/T), ..., \cos(2\pi k t/T), \sin(2\pi k t/T), ...\}$. This is the operation carried out by a sound equalizer. Choose some range of the signal you want to eliminate. Determine the appropriate projection matrix in Octave, and then reconstruct the signal formed from the filtered components by summing the resulting, filtered trigonometric series. The objectives are purposely stated without specifying all formulas; at this stage you should be able to translate the above into an algorithm. Keep in mind that the solution is very concise, no more than 5-10 lines of Octave code, so the important aspect is to carefully consider each operation needed to complete this task.

Solution. Suppose we keep frequencies from $n_1 = 100$ to $n_2 = 200$. Construct the matrix, find $\boldsymbol{Q}\boldsymbol{R}$ factorization and the projector $\boldsymbol{P} = \boldsymbol{Q}\boldsymbol{Q}^T$

```
octave> n1=100; n2=200; A=[C(:,n1:n2) S(:,n1:n2)];
octave> [Q,R]=qr(A,'0');
octave> z=Q*(Q'*y);
octave> plot(t,y,'k',t,z,'r'); xlabel('t'); ylabel('y, z');
octave> print -dpng HW3Fig3.png
```

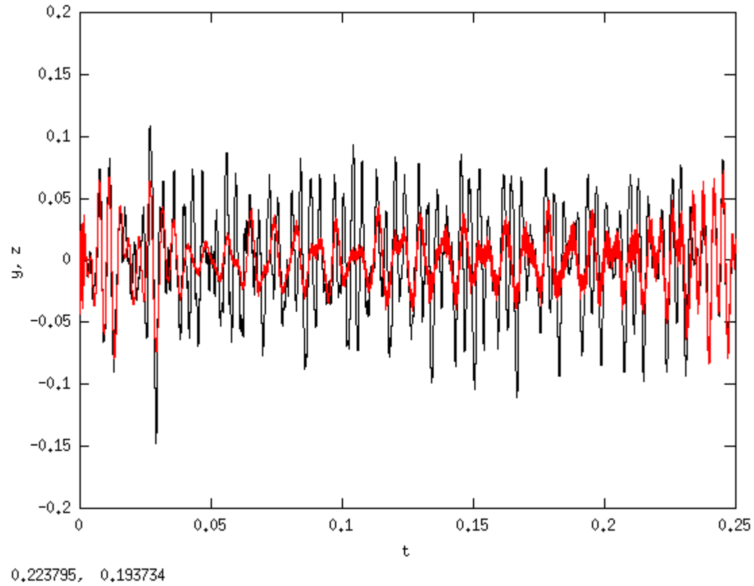

Figure 3. Original signal (black), and filtered signal (red) (frequencies 100 to 200). Notice that the filtered signal follows the overall pattern, but does not have higher frequencies.