# HW6 8.1.1.16

a) Find eigenvalues of $A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \in \mathbb{R}^{m \times m}$ $m = 5$

A diagonalizable

$\det(A) = \det(X \Lambda X^{-1}) = \det X \det \Lambda \det X^{-1} = \det \Lambda$

(2) $AX = X\Lambda$ $\det(AX) = \det(X\Lambda) \Rightarrow \det A \det X = \det X \det \Lambda$

(E.C.: find mistake in reasoning of (2))

---

Since A singular $\lambda = 0$ eigenvalue.

(Observe that for $m = 1$ $A = (1)$

$m = 2$ $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ $\lambda_1 = 0$ $\lambda_2 = 2$

$A - \lambda I = \begin{pmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{pmatrix}$

$\lambda = 2 \Rightarrow A - 2I = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$

$m = 3$ $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ $\lambda_1 = 0$

$\det|A - \lambda I| = \begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} = \begin{vmatrix} 3-\lambda & 1 & 1 \\ 3-\lambda & 1-\lambda & 1 \\ 3-\lambda & 1 & 1-\lambda \end{vmatrix}$

$= (3-\lambda) \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} = (3-\lambda) \begin{vmatrix} 1 & 1 & 1 \\ 0 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{vmatrix}$

---

$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1 & 1 & 1 & 1 \\ 1 & 1-\lambda & 1 & 1 & 1 \\ 1 & 1 & 1-\lambda & 1 & 1 \\ 1 & 1 & 1 & 1-\lambda & 1 \\ 1 & 1 & 1 & 1 & 1-\lambda \end{vmatrix} = \begin{vmatrix} 5-\lambda & 1 & 1 & 1 & 1 \\ 5-\lambda & 1-\lambda & 1 & 1 & 1 \\ 5-\lambda & 1 & 1-\lambda & 1 & 1 \\ 5-\lambda & 1 & 1 & 1-\lambda & 1 \\ 5-\lambda & 1 & 1 & 1 & 1-\lambda \end{vmatrix}$

$= (5-\lambda) \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1-\lambda & 1 & 1 & 1 \\ 1 & 1 & 1-\lambda & 1 & 1 \\ 1 & 1 & 1 & 1-\lambda & 1 \\ 1 & 1 & 1 & 1 & 1-\lambda \end{vmatrix} = (5-\lambda)(\lambda)^4$

$\lambda_1 = 0$ with algebraic multiplicity 4 ; geometric multiplicity 4

$\lambda_2 = 5$ — " — 1 ; geometric multiplicity 1

$\lambda_2 = 5$ ← 1 ———— 1 ; geometric multiplicity 1

$$A = \begin{pmatrix} 1&1&1&1&1\\ 1&1&1&1&1\\ 1&1&1&1&1\\ 1&1&1&1&1\\ 1&1&1&1&1 \end{pmatrix} \overset{\text{rref}}{\sim} \begin{pmatrix} 1&1&1&1&1\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0 \end{pmatrix}$$

$$N(A) = \left\langle \begin{pmatrix}1\\-1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\-1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\\-1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\\0\\-1\end{pmatrix} \right\rangle$$

$\dim N(A) = 4$

b) Find eigenvalues of $B = \begin{pmatrix} 3&1&1&1&1\\ 1&3&1&1&1\\ 1&1&3&1&1\\ 1&1&1&3&1\\ 1&1&1&1&3 \end{pmatrix}$

$B = A + 2I$ $\qquad \lambda_1^A = 0 \qquad \lambda_1^B = 2$

$\qquad \lambda_2^A = 5 \qquad \lambda_2^B = 7$

$$p_B(\lambda) = \det(B - \lambda I) = \det(A - (\lambda - 2) I)$$

for $x$ satisfying $Ax = \lambda x$ compute $(A + 2I)x = Ax + 2x = (\lambda + 2)x \Rightarrow$

$$Bx = (\lambda + 2)x$$

c) $\det B = 2^4 \, 7$

8.1.24. $A = \begin{pmatrix} 0&0&0&1\\ 0&0&1&0\\ 0&1&0&0\\ 1&0&0&0 \end{pmatrix}$

$P$ permutation matrix $\qquad \underline{P}\,\underline{P}^{-1} = I$

$$Ax = \lambda x$$

$$PAx = \lambda Px$$

$I \in \mathbb{R}^{4\times 4}$ has eigenbasis $X = I = (e_1 \; e_2 \; e_3 \; e_4)$

$P = A$ eigenvectors of $A$ $\quad PX = (Pe_1 \;\; Pe_2 \;\; Pe_3 \;\; Pe_4)$

8.3.1. Find SVD of

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix} \in \mathbb{R}^{2\times 2}$$

The SVD of $A = U \Sigma V^T$ with $U, V \in \mathbb{R}^{2\times 2}$ orthogonal

$$\Sigma = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

$$A A^T = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$$

$\det(AA^T - \lambda I) = (1-\lambda)(4-\lambda) \Rightarrow \lambda_1 = \sigma_1^2 = 4 \Rightarrow \sigma_1 = 2$

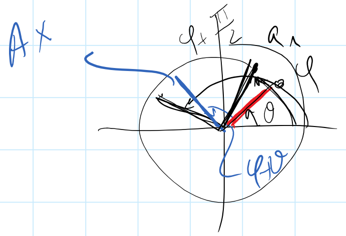
$\lambda_2 = \sigma_2^2 = 1 \Rightarrow \sigma_1 = 1$

$$\begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{U} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \underbrace{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}_{V^T}$$

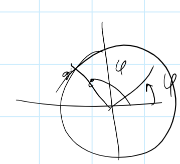
$$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -2 \\ 1 & 0 \end{pmatrix}$$

8.3.2. $A$ orthogonal $2\times 2$ matrix.

Use the image of the unit circle to find the singular values of $A$




$$\cancel{x} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$



$A = (a_1 \;\; a_2)$ $\qquad a_1^T \cdot a_2 = 0$

$a_1^T \cdot a_1 = a_2^T \cdot a_2 = 1$

$$A = (a_1 \quad a_2) \qquad a_1 \cdot a_2 = 0$$

$$a_1^T \cdot a_1 = a_2^T \cdot a_2 = 1$$

$$A x = \cos\theta \, a_1 + \sin\theta \, a_2$$

$$= \cos\theta \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix} + \sin\theta \begin{pmatrix} \cos\left(\varphi + \frac{\pi}{2}\right) = -\sin\varphi \\ \sin\left(\varphi + \frac{\pi}{2}\right) = \cos\varphi \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta\cos\varphi - \sin\theta\sin\varphi \\ \cos\theta\sin\varphi + \sin\theta\cos\varphi \end{pmatrix} = \begin{pmatrix} \cos(\varphi+\theta) \\ \sin(\varphi+\theta) \end{pmatrix}$$

(In complex

$$x = e^{i\theta}$$

rotation $e^{i\varphi} e^{i\theta} = e^{i(\varphi+\theta)}$ )

---

8.3.3. $A \in \mathbb{R}^{m \times m}$ orthogonal

singular values of $A$?

$$AA^T = \underline{I}$$

$$A^T A = \begin{pmatrix} a_1^T \\ \vdots \\ a_m^T \end{pmatrix} (a_1 \quad a_m) = \begin{pmatrix} a_1^T a_1 & a_1^T a_2 \\ & \end{pmatrix} = \underline{I}$$

8.3.4. Find singular values of $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

$$A A^T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

$$\det(AA^T - \lambda I) = \begin{vmatrix} 2-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = 1 - 3\lambda + \lambda^2$$

$$\lambda_{1,2} = \frac{3 \pm \sqrt{9-4}}{\ } = \frac{3+\sqrt{5}}{2}, \ \frac{3-\sqrt{5}}{\ }$$

$$\lambda_{1,2} = \frac{3 \pm \sqrt{9-4}}{2} = \begin{cases} \frac{3+\sqrt{5}}{2} \\ \frac{3-\sqrt{5}}{2} \end{cases}$$

$$\sigma_1 = \sqrt{\frac{3+\sqrt{5}}{2}} \qquad \sigma_1 \geq \sigma_2$$

$$\sigma_2 = \sqrt{\frac{3-\sqrt{5}}{2}}$$

---

8.3.17. $A \in \mathbb{R}^{m\times n}$ $\quad \operatorname{rank}(A) = n$

$$A = U\Sigma V^T$$

$$= \underset{\mathbb{R}^{m\times m}}{\underset{\cap}{U}} \quad \underset{\mathbb{R}^{m\times n}}{\underset{\cap}{\Sigma}} \quad \underset{\mathbb{R}^{n\times n}}{\underset{\cap}{V^T}}$$

$$= \begin{pmatrix} u_1 & u_2 & \dots & u_m \end{pmatrix} \begin{pmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_n \end{pmatrix} \begin{pmatrix} v_1^T \\ v_2^T \\ \vdots \\ v_n^T \end{pmatrix}$$

$$= \begin{pmatrix} u_1 & u_2 & \dots & u_m \end{pmatrix} \begin{pmatrix} \sigma_1 v_1^T \\ \sigma_2 v_2^T \\ \vdots \\ \sigma_n v_n^T \\ 0 \\ \vdots \end{pmatrix}$$

$$= \sigma_1 u_1 v_1^T + \dots + \sigma_n u_n v_n^T = \sum_{i=1}^{n} \sigma_i u_i v_i^T$$

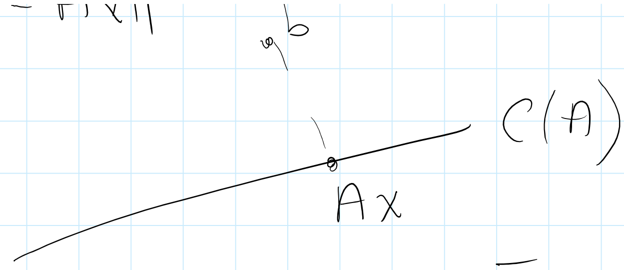
8.3.12.

$$A = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} 3 & -4 \\ 4 & 3 \end{pmatrix} \frac{1}{5}$$

Find least sq. solution of ~~$Ax = b$~~ $\quad$ Yuck! $\qquad b = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$

$$\min_{x \in \mathbb{R}^n} \| b - Ax \|$$

$\min_{x\in\mathbb{R}^n} \|b - Ax\|$




$A = U\Sigma V^T$

$C(A) = C(\hat{U})$

$P = \hat{U}\hat{U}^T$

$$\hat{U} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & -1 \\ 1 & -1 \end{pmatrix}$$

<u>V. imp.</u>: Many exercises just verify mastery of the <u>definitions</u>