MATH547 Extra Credit problems

(EC5, analytical) Consider $\boldsymbol{A} \in \mathbb{R}^{m \times m}$, $\boldsymbol{A} = \boldsymbol{A}^T$, for which we know that the eigenvalues are distinct and positive, $\lambda_1 > \lambda_2 > ... > \lambda_m > 0$, $\boldsymbol{A}\boldsymbol{x}_j = \lambda_j \boldsymbol{x}_j$. Consider the sequence of vectors $\boldsymbol{v}_k = \boldsymbol{A}^k \boldsymbol{v}_0$.

(1) What is $\lim_{k \to \infty} \boldsymbol{v}_k$?

(2) Based on the above analysis, devise a procedure to find the $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3$ as limits of sequences $\boldsymbol{v}_k = \boldsymbol{A}^k \boldsymbol{v}_0$ with appropriately chosen initial vectors $\boldsymbol{v}_0$.

(EC6, computational) In an experimental data acquisition process n measurements of m components are obtained, forming a matrix of observations $\boldsymbol{X} = (\ \boldsymbol{x}_1\ \ ...\ \ \boldsymbol{x}_n\ ) \in \mathbb{R}^{m \times n}$. Each column is an observation of the m components measured experimentally. The mean of the observations is

$$\boldsymbol{\mu} = \frac{1}{n} \sum_{j=1}^{n} \boldsymbol{x}_j \in \mathbb{R}^m.$$

The deviation from the mean of observation j is

$$\boldsymbol{y}_j = \boldsymbol{x}_j - \boldsymbol{\mu},$$

from which the centered observation matrix $\boldsymbol{Y} = (\ \boldsymbol{y}_1\ \ ...\ \ \boldsymbol{y}_n\ ) \in \mathbb{R}^{m \times n}$ is formed. The sample covariance matrix is defined as

$$\boldsymbol{S} = \frac{1}{n-1} \boldsymbol{Y}\boldsymbol{Y}^T \in \mathbb{R}^{m \times m}$$

(1) Express $\boldsymbol{S}$ in terms of the singular value decomposition of $\boldsymbol{Y}$.

(2) Revisit the yalefaces database. Consider each facial image as an observation $\boldsymbol{x}_j$. Display the principal components of the images, i.e., from

$$\boldsymbol{Y} = \boldsymbol{U}\boldsymbol{\Sigma}\mathbf{V}^T = \sum_{j=1}^{r} \sigma_j \boldsymbol{u}_j \boldsymbol{v}_j^T \cong \sum_{j=1}^{k} \sigma_j \boldsymbol{u}_j \boldsymbol{v}_j^T$$

show the images obtained for $k = 1, 2, ..., 6$.