Homework 1 Solution

Due date: February 5, 2015, 11:55PM.

Bibliography: Course lecture notes Lessons 4-8. Textbook pp. 77-121, Sections 2.1-2.5.

Carefully read the model solution. Take note both of the application of course concepts to solve the problems and the formatting of the answers. Strive to draft future homework solutions using this model.

1. (1 course point) Textbook p.87, Exercise 2.2.2. Which of the following are subspaces of $\mathbb{R}^3$? Justify your answer.

Solution. *General remarks*. The vector set $\mathcal{U}$ is a subspace of vector space $\mathcal{V}$, denoted as $\mathcal{U}\leq\mathcal{V}$, if for $\forall \boldsymbol{u},\boldsymbol{v}\in\mathcal{U}$, and any scalars $\alpha,\beta\in\mathcal{S}$ it holds that:

i. $\boldsymbol{u}\in\mathcal{V}$ (inclusion), and,

ii. (ii) $\alpha\boldsymbol{u}+\beta\boldsymbol{v}\in\mathcal{U}$ (closure) [Lesson 7, p.4, Textbook p. 86].

For this problem $\mathcal{V}=\mathbb{R}^3$, $\mathcal{S}=\mathbb{R}$. In particular for $\alpha=\beta=0$, $\forall\boldsymbol{u},\boldsymbol{v}\in\mathcal{U}$, $0\cdot\boldsymbol{u}+0\cdot\boldsymbol{v}=\boldsymbol{0}$, hence $\mathcal{U}$ is a vector subspace only if $\boldsymbol{0}\in\mathcal{U}$. In all exercises below, the vector set $\mathcal{U}$ is assumed to contain 3-component vectors, hence $\mathcal{U}\subseteq\mathbb{R}^3$, and the inclusion property is satisfied. The answers verify closure to prove $\mathcal{U}\leq\mathcal{V}$, or from $\boldsymbol{0}\notin\mathcal{U}$ show that $\mathcal{U}$ is not a vector subspace.

a) $\mathcal{U}_a=\{(x,y,z)^T\,|\,x+y+z+1=0\}$. No, $\mathcal{U}_a\not\leq\mathbb{R}^3$, since $(x,y,z)^T=\boldsymbol{0}$ does not satisfy $x+y+z+1=0$, hence $\boldsymbol{0}\notin\mathcal{U}_a$.

b) $\mathcal{U}_b=\{(t,-t,0)^T\,|\,t\in\mathbb{R}\}$. For arbitrary $\boldsymbol{u}=(s,-s,0)^T\in\mathcal{U}_b$, $\boldsymbol{v}=(t,-t,0)^T\in\mathcal{U}_b$, $\alpha,\beta\in\mathbb{R}$, compute

$$\alpha\boldsymbol{u}+\beta\boldsymbol{v}=\alpha\begin{pmatrix}s\\-s\\0\end{pmatrix}+\beta\begin{pmatrix}t\\-t\\0\end{pmatrix}=\begin{pmatrix}\alpha s+\beta t\\-(\alpha s+\beta t)\\0\end{pmatrix}=\begin{pmatrix}w\\-w\\0\end{pmatrix}\in\mathcal{U}_b,$$

with $w=\alpha s+\beta t\in\mathbb{R}$, hence, yes, $\mathcal{U}_b\leq\mathbb{R}^3$.

c) $\mathcal{U}_c=\{(r-s,r+2s,-s)^T|r,s\in\mathbb{R}\}$. For arbitrary $\boldsymbol{f}=(r-s,r+2s,-s)^T\in\mathcal{U}_c$, $\boldsymbol{g}=(t-u,t+2u,-t)^T\in\mathcal{U}_c$, $\alpha,\beta\in\mathbb{R}$, compute

$$\alpha\boldsymbol{f}+\beta\boldsymbol{g}=\alpha\begin{pmatrix}r-s\\r+2s\\-s\end{pmatrix}+\beta\begin{pmatrix}t-u\\t+2u\\-u\end{pmatrix}=\begin{pmatrix}\alpha r+\beta t-(\alpha s+\beta u)\\\alpha r+\beta t+2(\alpha s+\beta u)\\-(\alpha s+\beta u)\end{pmatrix}=\begin{pmatrix}v-w\\v+2w\\-w\end{pmatrix}\in\mathcal{U}_c,$$

with $v=\alpha r+\beta t\in\mathbb{R}$, $w=\alpha s+\beta u\in\mathbb{R}$, hence, yes, $\mathcal{U}_c\leq\mathbb{R}^3$.

d) $\mathcal{U}_d=\{(0,s,t)^T\,|\,s,t\in\mathbb{R}\}$. For arbitrary $\boldsymbol{f}=(0,r,s)^T\in\mathcal{U}_d$, $\boldsymbol{g}=(0,t,u)^T\in\mathcal{U}_d$, $\alpha,\beta\in\mathbb{R}$, compute

$$\alpha\boldsymbol{f}+\beta\boldsymbol{g}=\alpha\begin{pmatrix}0\\r\\s\end{pmatrix}+\beta\begin{pmatrix}0\\t\\u\end{pmatrix}=\begin{pmatrix}0\\\alpha r+\beta t\\\alpha s+\beta u\end{pmatrix}=\begin{pmatrix}0\\v\\w\end{pmatrix}\in\mathcal{U}_d,$$

with $v=\alpha r+\beta t\in\mathbb{R}$, $w=\alpha s+\beta u\in\mathbb{R}$, hence, yes, $\mathcal{U}_d\leq\mathbb{R}^3$.

e) $\mathcal{U}_e=\{(r,s,1)^T|r,s\in\mathbb{R}\}$. No, $\mathcal{U}_e\not\leq\mathbb{R}^3$, since $\boldsymbol{0}\notin\mathcal{U}_e$.

f) $\mathcal{U}_f=\{(x,y,z)^T|x,y,z\in\mathbb{R},\text{and}\,x\geqslant y\geqslant z\}$. No, counterexample with $\boldsymbol{f},\boldsymbol{g}\in\mathcal{U}_f$.

$$1\cdot\boldsymbol{f}+(-2)\cdot\boldsymbol{g}=1\cdot\begin{pmatrix}2\\1\\0\end{pmatrix}+(-2)\cdot\begin{pmatrix}4\\2\\0\end{pmatrix}=\begin{pmatrix}-2\\-1\\0\end{pmatrix}\notin\mathcal{U}_f.$$

g) $\mathcal{U}_g = \{(x,y,z)^T | x-y=z\}$. For arbitrary $\boldsymbol{f} = (s,t,s-t)^T \in \mathcal{U}_g$, $\boldsymbol{g} = (u,v,u-v)^T \in \mathcal{U}_g$, $\alpha, \beta \in \mathbb{R}$, compute

$$\alpha \boldsymbol{f} + \beta \boldsymbol{g} = \alpha \begin{pmatrix} s \\ t \\ s-t \end{pmatrix} + \beta \begin{pmatrix} u \\ v \\ u-v \end{pmatrix} = \begin{pmatrix} \alpha s + \beta u \\ \alpha t + \beta v \\ \alpha s + \beta u - (\alpha t + \beta v) \end{pmatrix} \in \mathcal{U}_d,$$

hence, yes, $\mathcal{U}_g \leq \mathbb{R}^3$.

h) $\mathcal{U}_g = \{(x,y,z)^T \,|\, z = xy\}$. No, counterexample with $\boldsymbol{f}, \boldsymbol{g} \in \mathcal{U}_f$.

$$1 \cdot \boldsymbol{f} + 1 \cdot \boldsymbol{g} = 1 \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + 1 \cdot \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 5 \end{pmatrix} \notin \mathcal{U}_g \text{ since } 5 \neq 3 \cdot 3.$$

i) $\mathcal{U}_i = \{(x,y,z)^T \,|\, x^2 + y^2 + z^2 = 0\}$. Yes, since $\mathcal{U}_i = \{\boldsymbol{0}\}$.

j) $\mathcal{U}_j = \{(x,y,z)^T \,|\, xy = yz = xz\}$. Solution of system leads to

$$\mathcal{U}_j = \{(x,0,0)^T | x \in \mathbb{R}\} \cup \{(0,y,0)^T | y \in \mathbb{R}\} \cup \{(0,0,z)^T | z \in \mathbb{R}\} \cup \{(x,x,x) | x \in \mathbb{R}\}.$$

$\mathcal{U}_j$ is not a vector subspace by counterexample

$$1 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 1 \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \in \mathcal{U}_j.$$

2. (1 course point) Textbook p.93, Exercises 2.3.4 and 2.3.7.

2.3.4. solution general remarks. The vector set $\mathcal{A} = \{\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n\}$ spans $\mathbb{R}^2$ if $n \geqslant 2$, and $\mathcal{A}$ contains 2 linearly independent vectors. Using matrix concepts, $C(\boldsymbol{A}) = \mathbb{R}^2$, if the matrix $\boldsymbol{A} = (\, \boldsymbol{a}_1 \;\; \boldsymbol{a}_2 \;\; \ldots \;\; \boldsymbol{a}_n \,) \in \mathbb{R}^{2 \times n}$ has 2 linearly independent column vectors. When $n = 2$, equivalently, the only solution to $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}$ is $\boldsymbol{x} = \boldsymbol{0}$ [Lesson 6, p.7]. The solution to $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}$ can be found by Gaussian elimination, calculated through bordered matrix $(\, \boldsymbol{A} \;\; \boldsymbol{0} \,)$.

a) $\boldsymbol{A}_a = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. No, since $n = 1 < 2$.

b) $\boldsymbol{A}_b = \begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix}$. From $(\, \boldsymbol{A}_b \;\; \boldsymbol{0} \,) = \begin{pmatrix} 2 & 1 & 0 \\ -1 & 3 & 0 \end{pmatrix} \sim \begin{pmatrix} 2 & 1 & 0 \\ 0 & 3.5 & 0 \end{pmatrix}$ find solution $\boldsymbol{x}_b = \boldsymbol{0}$, hence, yes, $C(\boldsymbol{A}_b) = \mathbb{R}^2$.

c) $\boldsymbol{A}_c = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$. From $(\, \boldsymbol{A}_c \;\; \boldsymbol{0} \,) = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \end{pmatrix} \sim \begin{pmatrix} 2 & -1 & 0 \\ -1 & 1.5 & 0 \end{pmatrix}$ find solution $\boldsymbol{x}_c = \boldsymbol{0}$, hence, yes, $C(\boldsymbol{A}_c) = \mathbb{R}^2$.

d) $\boldsymbol{A}_d = \begin{pmatrix} 6 & -4 \\ -9 & 6 \end{pmatrix}$. From $(\, \boldsymbol{A}_d \;\; \boldsymbol{0} \,) = \begin{pmatrix} 6 & -4 & 0 \\ -9 & 6 & 0 \end{pmatrix} \sim \begin{pmatrix} 6 & -4 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ find solution $\boldsymbol{x}_d = (\, 0 \;\; 1 \,)^T \neq \boldsymbol{0}$, hence, no, $C(\boldsymbol{A}_c) \neq \mathbb{R}^2$.

e) $\boldsymbol{A}_e = \begin{pmatrix} 1 & 2 & 3 \\ -1 & -1 & -1 \end{pmatrix}$. From $(\, (\, \boldsymbol{a}_1 \;\; \boldsymbol{a}_2 \,) \;\; \boldsymbol{0} \,) = \begin{pmatrix} 1 & 2 & 0 \\ -1 & -1 & 0 \end{pmatrix} \sim \begin{pmatrix} 2 & 2 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ find solution $\boldsymbol{x}_e = \boldsymbol{0}$, hence, yes, $C(\boldsymbol{A}_e) = \mathbb{R}^2$.

f) No, since $\boldsymbol{a}_3 = 2\boldsymbol{a}_2$, and $\boldsymbol{a}_1 = \boldsymbol{0}$.

2.3.7. solution. Define

$$\mathcal{S} = \left\{ \begin{pmatrix} a & c \\ c & b \end{pmatrix} | a,b,c \in \mathbb{R} \right\}, \boldsymbol{U}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \boldsymbol{U}_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \boldsymbol{U}_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

From identity

$$\begin{pmatrix} a & c \\ c & b \end{pmatrix} = a\boldsymbol{U}_1 + b\boldsymbol{U}_2 + c\boldsymbol{U}_3,$$

deduce $\operatorname{span}\{\boldsymbol{U}_1, \boldsymbol{U}_2, \boldsymbol{U}_3\} = \mathcal{S}$.

3. (1 course point) Textbook p.94, Exercises 2.3.8 and 2.3.9.

2.3.8 solution.

a) $\mathcal{P}^{(2)} = \{x_0 + x_1 t + x_2 t^2 | x_0, x_1, x_2 \in \mathbb{R}\}$. The coordinates of $p \in \mathcal{P}^{(2)}$ are $(\, x_0 \; x_1 \; x_2 \,)^T$, which is denoted as $p \to (\, x_0 \; x_1 \; x_2 \,)^T$, and

$$x^2 + 1 \to \boldsymbol{a}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, x^2 - 1 \to \boldsymbol{a}_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, x^2 + x + 1 \to \boldsymbol{a}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Verifying that $\text{span}\{x^2 + 1, x^2 - 1, x^2 + x + 1\}$ covers $\mathcal{P}^{(2)}$ is equivalent to verifying $C(\boldsymbol{A}) = \mathbb{R}^3$, with $\boldsymbol{A} = (\, \boldsymbol{a}_1 \; \boldsymbol{a}_2 \; \boldsymbol{a}_3 \,)$, or that the only solution to $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}$ is $\boldsymbol{x} = \boldsymbol{0}$. Carry out Gauss elimination

$$(\, \boldsymbol{A} \;\; \boldsymbol{0} \,) = \begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \Rightarrow \boldsymbol{x} = \boldsymbol{0},$$

hence, yes, $\text{span}\{x^2 + 1, x^2 - 1, x^2 + x + 1\} = \mathcal{P}^{(2)}$.

b) Proceed as above,

$$(\, \boldsymbol{A} \;\; \boldsymbol{0} \,) = \begin{pmatrix} 1 & -1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \Rightarrow \boldsymbol{x} = \boldsymbol{0},$$

hence, yes, $\text{span}\{1, -1 + x, 1 + x^2, -1 + x^3\} = \mathcal{P}^{(3)}$.

c) Proceed as above,

$$(\, \boldsymbol{A} \;\; \boldsymbol{0} \,) = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \Rightarrow \boldsymbol{x} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \neq \boldsymbol{0},$$

is a nontrivial solution, so columns are not independent and, no, $\text{span}\{1, -1 + x, 1 + x^2, -1 + x^3\} \neq \mathcal{P}^{(3)}$.

2.3.9. solution. Denote $\mathcal{F} = \text{span}\{1, x, \sin x, \sin^2 x\} = \text{span}\{u_1(x), u_2(x), u_3(x), u_4(x)\}$. To solve this problem, either explicitly find the coefficients of the linear combination, or show from choice of particular evaluation points x that no such linear combination is possible.

a) Yes, $f_a(x) = 3 - 5x = 3u_1(x) + 5u_2(x) \in \mathcal{F}$.

b) For $f_b(x) = x^2 + \sin^2 x \in \mathcal{F}$, there must exist a_1, a_2, a_3, a_4 such that $f_b(x) = a_1 u_1(x) + a_2 u_2(x) + a_3 u_3(x) + a_4 u_4(x)$ for all x. At $x = 0, x = \pi/2$

$$f_b(0) = 0 = a_1 \cdot u_1(0) \Rightarrow a_1 = 0$$
$$f_b\left(\frac{\pi}{2}\right) = \left(\frac{\pi}{2}\right)^2 = a_2 \frac{\pi}{2} \Rightarrow a_2 = \frac{\pi}{2},$$

but at $x = -\pi/2$

$$f_b\left(-\frac{\pi}{2}\right) = \left(\frac{\pi}{2}\right)^2 \neq -\left(\frac{\pi}{2}\right)^2 = a_2 \cdot \left(-\frac{\pi}{2}\right),$$

hence, no, $f_b(x) \notin \mathcal{F}$.

c) $f_c(x) = \sin x - 2\cos x$. At $x = 0, x = \pi$

$$f_c(0) = -2 = a_1 \Rightarrow a_1 = -2$$
$$f_c(\pi) = 2 = -2 + a_2 \pi \Rightarrow a_2 = \frac{4}{\pi},$$

but at $x = 2\pi$

$$f_c(2\pi) = -2 \neq -2 + \frac{4}{\pi} 2\pi = 2,$$

hence, no, $f_c(x) \notin \mathcal{F}$.

d) $f_d(x)=\cos x$. At $x=0, x=\pi$

$$f_d(0)=1=a_1 \Rightarrow a_1=1$$
$$f_d(\pi)=-1=1+a_2\pi \Rightarrow a_2=-\frac{2}{\pi},$$

but at $x=2\pi$

$$f_d(2\pi)=1\neq 1-\frac{2}{\pi}2\pi=-3.$$

hence, no, $f_d(x)\notin\mathcal{F}$.

e) $f_e(x)=x\sin x$. At $x=0, x=\pi/2$

$$f_e(0)=0=a_1 \Rightarrow a_1=0$$
$$f_e\left(\frac{\pi}{2}\right)=\frac{\pi}{2}=a_2\frac{\pi}{2}\Rightarrow a_2=1,$$

but at $x=\pi$

$$f_e(\pi)=0\neq 1\cdot\pi=\pi.$$

hence, no, $f_e(x)\notin\mathcal{F}$.

f) $f_f(x)=e^x$. At $x=0, x=\pi$

$$f_f(0)=1=a_1 \Rightarrow a_1=1$$
$$f_f(\pi)=e^\pi=a_1+a_2\pi \Rightarrow a_2=\frac{e^\pi-1}{\pi},$$

but at $x=2\pi$

$$f_f(\pi)=e^{2\pi}\neq 1+\frac{e^\pi-1}{\pi}2\pi=2e^\pi-1.$$

hence, no, $f_f(x)\notin\mathcal{F}$.

4. (1 course point) Textbook p.105, Exercises 2.4.2, 2.4.3, and 2.4.4.

Solution.

2.4.2. A basis is a linearly independent spanning set. The vectors sets are basis of $\mathbb{R}^3$ if they contain 3 linearly independent vectors.

a) No, only 2 vectors.

b)

$$\left(\begin{array}{cc}\boldsymbol{A} & \boldsymbol{0}\end{array}\right)=\left(\begin{array}{cccc}0 & -1 & 1 & 0\\ 1 & 3 & 3 & 0\\ -5 & 0 & 0 & 0\end{array}\right)\sim\left(\begin{array}{cccc}1 & 3 & 3 & 0\\ 0 & -1 & 1 & 0\\ 0 & 15 & 15 & 0\end{array}\right)\sim\left(\begin{array}{cccc}1 & 3 & 3 & 0\\ 0 & -1 & 1 & 0\\ 0 & 0 & 30 & 0\end{array}\right)\Rightarrow\boldsymbol{x}=\boldsymbol{0},$$

yes, columns of $\boldsymbol{A}$ are a basis.

c)

$$\left(\begin{array}{cc}\boldsymbol{A} & \boldsymbol{0}\end{array}\right)=\left(\begin{array}{cccc}0 & -1 & 1 & 0\\ 4 & 0 & -8 & 0\\ -1 & 1 & 1 & 0\end{array}\right)\sim\left(\begin{array}{cccc}-1 & 1 & 1 & 0\\ 0 & -1 & 1 & 0\\ 0 & 4 & -4 & 0\end{array}\right)\sim\left(\begin{array}{cccc}-1 & 1 & 1 & 0\\ 0 & -1 & 1 & 0\\ 0 & 0 & 0 & 0\end{array}\right)\Rightarrow\boldsymbol{x}=\left(\begin{array}{c}0\\0\\1\end{array}\right)\neq\boldsymbol{0},$$

is a non-trivial solution so, no, columns of $\boldsymbol{A}$ are not a basis.

d) No, more vectors than dimension of space.

2.4.3.

a) The plane is of dimension 2. Two linearly independent vectors within the plane $z-2y=0$ are

$$\boldsymbol{u}_1=\left(\begin{array}{c}1\\0\\0\end{array}\right),\boldsymbol{u}_2=\left(\begin{array}{c}0\\1\\2\end{array}\right).$$

b) The plane is of dimension 2. Two linearly independent vectors within the plane $4x+3y-z=0$ are

$$\boldsymbol{u}_1=\left(\begin{array}{c}1\\1\\7\end{array}\right),\boldsymbol{u}_2=\left(\begin{array}{c}0\\1\\3\end{array}\right).$$

c) The hyperplane is of dimension 3. Three linearly independent vectors within the hyperplane $x+2y+z-w=0$ are

$$\boldsymbol{u}_1=\begin{pmatrix}1\\0\\0\\1\end{pmatrix}, \boldsymbol{u}_2=\begin{pmatrix}0\\1\\0\\2\end{pmatrix}, \boldsymbol{u}_3=\begin{pmatrix}0\\0\\1\\1\end{pmatrix}.$$

2.4.4. Let

$$\boldsymbol{V}=(\ \boldsymbol{v}_1\ \ \boldsymbol{v}_2\ \ \boldsymbol{v}_3\ \ \boldsymbol{v}_4\ )=\begin{pmatrix}1 & 3 & 2 & 4\\ 0 & -1 & -1 & -1\\ 2 & 1 & -1 & 3\end{pmatrix}.$$

a) Since reduction to row echelon form

$$(\ \boldsymbol{v}_1\ \ \boldsymbol{v}_2\ \ \boldsymbol{v}_3\ \ \boldsymbol{v}_4\ )=\begin{pmatrix}1 & 3 & 2 & 4\\ 0 & -1 & -1 & -1\\ 2 & 1 & -1 & 3\end{pmatrix}\sim\begin{pmatrix}1 & 3 & 2 & 4\\ 0 & -1 & -1 & -1\\ 0 & -5 & -5 & -5\end{pmatrix}\sim\begin{pmatrix}1 & 3 & 2 & 4\\ 0 & 1 & 1 & 1\\ 0 & 0 & 0 & 0\end{pmatrix},$$

exhibits only 2 pivots, deduce $\text{rank}(\boldsymbol{V})=2$, so the vectors do not span $\mathbb{R}^3$.

b) No, $\{\boldsymbol{v}_1,\boldsymbol{v}_2,\boldsymbol{v}_3,\boldsymbol{v}_4\}$ are not linearly independent, from above rref.

c) No, $\{\boldsymbol{v}_1,\boldsymbol{v}_2,\boldsymbol{v}_3,\boldsymbol{v}_4\}$ is not a basis since it does not span $\mathbb{R}^3$.

d) From rref $\dim \operatorname{span}\{\boldsymbol{v}_1,\boldsymbol{v}_2,\boldsymbol{v}_3,\boldsymbol{v}_4\}=2$.

5. (Computer application 4 course points) Textbook pp.105-106. Solve Exercises, 2.4.1, 2.4.2, 2.4.9 computationally by use of Octave.

Solution. Answers to all questions are available from reduction to row echelon form.

2.4.1.

a) Yes, 2 pivot columns.

```
octave> A=[1 -3; -2 5]'; disp(rref(A));
        1   0
        0   1
octave>
```

b) No, 1 pivot column.

```
octave> A=[1 -1; -1 1]'; disp(rref(A));
        1  -1
        0   0
octave>
```

c) Yes, 2 pivots.

```
octave> A=[1 2; 2 1]'; disp(rref(A));
        1   0
        0   1
octave>
```

d) No, zero vector.

e) No, more vectors than dimension of $\mathbb{R}^2$.

```
octave>
```

2.4.2.

a) No, only 2 vectors.

b) Yes, 3 pivots.

```
octave> A=[0 1 -5; -1 3 0; 1 3 0]'; disp(rref(A));
       1  -0   0
       0   1   0
       0   0   1
octave>
```

c) No, 2 pivots.

```
octave> A=[0 4 -1; -1 0 1; 1 -8 1]'; disp(rref(A));
       1   0  -2
       0   1  -1
       0   0   0
octave>
```

d) No, more vectors than dimension.

2.4.9.

a) $\dim C(\boldsymbol{A}) = 1$, and a basis is first column $\boldsymbol{a}_1$

```
octave> A=[3 1 -1; -6 -2 2]'; disp(rref(A));
       1  -2
       0   0
       0   0
octave>
```

b) $\dim C(\boldsymbol{A}) = 2$, and the first two columns form a basis

```
octave> A=[2 0 1; 0 -1 3; 2 1 -2]'; disp(rref(A));
       1   0   1
       0   1  -1
       0   0   0
octave> disp(rref(A(:,1:2)));
       1   0
       0   1
       0   0
octave>
```

c) $\dim C(\boldsymbol{A}) = 3$, and columns 1,2,4 form a basis.

```
octave> A=[1 0 -1 2; 0 1 1 3; 2 -1 -3 1; 1 -2 1 1]'; disp(rref(A));
       1   0   2   0
       0   1  -1   0
       0   0   0   1
       0   0   0   0
octave> disp(rref([A(:,1:2) A(:,4)]));
       1   0   0
       0   1   0
       0   0   1
       0   0   0
octave>
```