Homework 5 solution

Due date: April 1, 2016, 11:55PM.

Bibliography: Course lecture notes Lessons 21-23. Textbook pp. 395-424, Sections 8.2-8.4.

Typical solution procedures are shown

1. (1 course point) Textbook p.400, Exercise 8.2.1

Solution. (a) Eigenvalue computation by finding roots of characteristic polynomial

$$p(\lambda)=\det(\boldsymbol{A}-\lambda\boldsymbol{I})=\left|\begin{array}{cc} 1-\lambda & -2\\ -2 & 1-\lambda \end{array}\right|=\lambda^2-2\lambda-3=(\lambda-3)(\lambda+1)=0\Rightarrow\lambda_1=-1,\lambda_2=3.$$

Eigenvector computation by finding basis for null spaces of $\boldsymbol{A}-\lambda\boldsymbol{I}$

$$\boldsymbol{A}-\lambda_1\boldsymbol{I}=\left(\begin{array}{cc} 2 & -2\\ -2 & 2 \end{array}\right)\sim\left(\begin{array}{cc} 2 & -2\\ 0 & 0 \end{array}\right)\Rightarrow\left\{\begin{array}{rcl} 2\cdot x_{11}-2\cdot x_{21} & = & 0\\ 0\cdot x_{11}+0\cdot x_{21} & = & 0 \end{array}\right.\Rightarrow\boldsymbol{x}_1=\alpha\left(\begin{array}{c} 1\\ 1 \end{array}\right);\|\boldsymbol{x}_1\|=1\Rightarrow\alpha=\frac{1}{\sqrt{2}}.$$

$$\boldsymbol{A}-\lambda_2\boldsymbol{I}=\left(\begin{array}{cc} -2 & -2\\ -2 & -2 \end{array}\right)\sim\left(\begin{array}{cc} 2 & 2\\ 0 & 0 \end{array}\right)\Rightarrow\left\{\begin{array}{rcl} 2\cdot x_{12}+2\cdot x_{22} & = & 0\\ 0\cdot x_{12}+0\cdot x_{22} & = & 0 \end{array}\right.\Rightarrow\boldsymbol{x}_2=\alpha\left(\begin{array}{c} -1\\ 1 \end{array}\right);\|\boldsymbol{x}_2\|=1\Rightarrow\alpha=\frac{1}{\sqrt{2}}.$$

(b) Once the above operations have been learned and become routine, one can use Octave to carry out the arithmetic

```
octave> A=[1 -2./3.; 1/2. -1/6.]; c=poly(A); disp(c);
    1.00000  -0.83333   0.16667
octave> lambda=roots(c); disp(lambda');
    0.50000   0.33333
octave> X=[ null(A-lambda(1)*eye(2)) null(A-lambda(2)*eye(2)) ]; disp(X);
    0.80000   0.70711
    0.60000   0.70711
octave> disp( inv(X)*A*X );
    0.50000   0.00000
    0.00000   0.33333
octave>
```

(c) The operations of finding eigenvalues and eigenvectors are frequently encountered, and there exist operations to directly obtain the eigensystem associated with a matrix.

```
octave> A=[3 1; -1 1]; [X,L]=eig(A); disp([X L]);
    0.70711  -0.70711   2.00000   0.00000
   -0.70711   0.70711   0.00000   2.00000
octave> det(A)
  4
octave>
```

Note, that interpreting the results requires understanding of theoretical concepts. In this case, the matrix has eigenvalue $\lambda=2$ with algebraic multiplicity $m_\lambda=2$. The geometric multiplicity is $n_\lambda=\dim(N(\boldsymbol{A}-\lambda\boldsymbol{I}))=1$, and the matrix is defective. Notice that Octave gives only one basis vector for the null space, and the rank of the $\boldsymbol{X}$ matrix returned by the eig procedure is equal to one

```
octave> disp(null(A-2*eye(2)));
   -0.70711
```

```
    0.70711
octave> disp( rank(X) );
   2
octave>
```

(j) This example is instructive since it shows how identifying a block structure in the matrix can simplify computations. Notice how block submatrices are identified, defined, and used

```
octave> A11=[3 4;4 3]; A12=zeros(2); A21=zeros(2); A22=[1 3;4 5];
octave> A=[A11 A12; A21 A22]; disp(A);
    3   4   0   0
    4   3   0   0
    0   0   1   3
    0   0   4   5
octave>
```

The eigenvalues could be found as the roots of the characteristic polynomial of the $\boldsymbol{A}$ matrix

```
octave> c=poly(A); r=roots(c); disp(r');
    7.0000 - 0.0000i   7.0000 - 0.0000i  -1.0000 - 0.0000i  -1.0000 + 0.0000i
octave>
```

In this particular case with $\boldsymbol{A}_{12} = \boldsymbol{A}_{21} = \mathbf{0}$, the characteristic polynomial can be computed as

$$\det(\boldsymbol{A} - \lambda \boldsymbol{I}_4) = \begin{vmatrix} \boldsymbol{A}_{11} - \lambda \boldsymbol{I}_2 & \mathbf{0} \\ \mathbf{0} & \boldsymbol{A}_{22} - \lambda \boldsymbol{I}_2 \end{vmatrix} = \det(\boldsymbol{A}_{11} - \lambda \boldsymbol{I}_2)\det(\boldsymbol{A}_{22} - \lambda \boldsymbol{I}_2).$$

Here we explicitly show the dimension of the identity matrix through a subscript.

```
octave> c1=poly(A11); r1=roots(c1); c2=poly(A22); r2=roots(c2);
octave> disp([r1 r2]);
    7   7
   -1  -1
octave>
```

The same roots are found by working with submatrices. This is typically advantageous since it reduces number of arithmetic operations.

(EC4.a 1 point) Determine the most general conditions on submatrices $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}$ such that

$$\det\begin{pmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{D} \end{pmatrix} = \det(\boldsymbol{A}\boldsymbol{D} - \boldsymbol{B}\boldsymbol{C}).$$

(k) In this example, the triangular structure of the matrix can be used to simplify computations. Indeed the eigenvalues are directly read off the diagonal

$$\det(\boldsymbol{A} - \lambda \boldsymbol{I}) = \begin{vmatrix} 4-\lambda & 0 & 0 & 0 \\ 1 & 3-\lambda & 0 & 0 \\ -1 & 1 & 2-\lambda & 0 \\ 1 & -1 & 1 & 1-\lambda \end{vmatrix} = (4-\lambda)(3-\lambda)(2-\lambda)(1-\lambda).$$

```
octave> A=[4 0 0 0; 1 3 0 0; -1 1 2 0; 1 -1 1 1]; disp(eig(A)');
    1   2   3   4
octave>
```

2. (1 course point) Textbook p.401, Exercise 8.2.7

(c) Procedures for complex-valued matrices are identical to those for real matrices

$$p(\lambda)=\det(\boldsymbol{A}-\lambda\boldsymbol{I})=\begin{vmatrix} i-\lambda & 1 \\ 0 & -1+i-\lambda \end{vmatrix}=(\lambda-i)(\lambda-i+1)=0\Rightarrow\lambda_1=i,\lambda_2=i-1.$$

$$\boldsymbol{A}-\lambda_1\boldsymbol{I}=\begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}\sim\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\Rightarrow\left\{\begin{array}{lll} 0\cdot x_{11}+1\cdot x_{21} & = & 0 \\ 0\cdot x_{11}+0\cdot x_{21} & = & 0 \end{array}\right.\Rightarrow\boldsymbol{x}_1=\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\boldsymbol{A}-\lambda_2\boldsymbol{I}=\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\Rightarrow\left\{\begin{array}{lll} 1\cdot x_{12}+1\cdot x_{22} & = & 0 \\ 0\cdot x_{12}+0\cdot x_{22} & = & 0 \end{array}\right.\Rightarrow\boldsymbol{x}_2=\alpha\begin{pmatrix} -1 \\ 1 \end{pmatrix};\|\boldsymbol{x}_2\|=1\Rightarrow\alpha=\frac{1}{\sqrt{2}}.$$

Octave computational procedures work equally well with complex numbers

```
octave> i=sqrt(-1); A=[i 1; 0 -1+i]; c=poly(A); disp(c);
     1 + 0i    1 - 2i   -1 - 1i
octave> r=roots(c); disp(r);
    -1.00000 + 1.00000i
     0.00000 + 1.00000i
octave> [X,L]=eig(A); disp(diag(L));
     0 + 1i
    -1 + 1i
octave> disp(X);
     1.00000    0.70711
     0.00000   -0.70711
```

3. (1 course point) Textbook p.404, Exercises 8.2.19-22

Solution. (19) It is given that $\boldsymbol{A}\boldsymbol{x}=\lambda\boldsymbol{x}$. (a) Multiply eigenvalue relation by c to obtain $(c\boldsymbol{A})\boldsymbol{x}=(c\lambda)\,\boldsymbol{x}$.

(b) Add $d\boldsymbol{x}$ to both sides to obtain $\boldsymbol{A}\boldsymbol{x}+d\boldsymbol{I}\boldsymbol{x}=\lambda\boldsymbol{x}+d\boldsymbol{x}\Rightarrow(\boldsymbol{A}+d\boldsymbol{I})=(\lambda+d)\boldsymbol{x}$, hence $\lambda+d$ is an eigenvalue of $\boldsymbol{A}+d\boldsymbol{I}$.

(c) Combine above.

(20) Multiply $\boldsymbol{A}\boldsymbol{x}=\lambda\boldsymbol{x}$, on the left by $\boldsymbol{A}$ to obtain $\boldsymbol{A}^2\boldsymbol{x}=\lambda\boldsymbol{A}\boldsymbol{x}=\lambda(\lambda\boldsymbol{x})=\lambda^2\boldsymbol{x}$.

(21) (a) False. The key observation here is that even if two matrices have the same eigenvalue, they generally have different eigenvectors, $\boldsymbol{A}\boldsymbol{x}=\lambda\boldsymbol{x}$ and $\boldsymbol{B}\boldsymbol{y}=\lambda\boldsymbol{y}$. Counter example

$$\lambda=0,\boldsymbol{A}=\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\boldsymbol{B}=\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},\boldsymbol{A}+\boldsymbol{B}=\boldsymbol{I}\text{ has }\lambda_{1,2}=1.$$

(b) True. Given $\boldsymbol{A}\boldsymbol{v}=\lambda\boldsymbol{v}$ and $\boldsymbol{B}\boldsymbol{v}=\mu\boldsymbol{v}$ it results from adding the two equalities that $(\boldsymbol{A}+\boldsymbol{B})\boldsymbol{v}=(\lambda+\mu)\boldsymbol{v}$.

(22) False. Again, the two matrices need not have the same eigenvectors. Counter example

$$\lambda=1,\boldsymbol{A}=\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\mu=1,\boldsymbol{B}=\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},\boldsymbol{A}\boldsymbol{B}=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\text{has eigenvalues }\{0,0\}.$$

4. (1 course point) Textbook p.404, Exercises 8.2.23-26

(23) Consider $(\boldsymbol{A}\boldsymbol{B})\boldsymbol{x}=\lambda\boldsymbol{x}$. Multiply on left by $\boldsymbol{B}$ to obtain $\boldsymbol{B}(\boldsymbol{A}\boldsymbol{B})\boldsymbol{x}=\lambda\boldsymbol{B}\boldsymbol{x}\Rightarrow(\boldsymbol{B}\boldsymbol{A})\boldsymbol{y}=\lambda\boldsymbol{y}$, proving $\boldsymbol{A}\boldsymbol{B}$ and $\boldsymbol{B}\boldsymbol{A}$ have same eigenvalues and eigenvectors related by $\boldsymbol{y}=\boldsymbol{B}\boldsymbol{x}$.

(24) (a) Multiply $\boldsymbol{A}\boldsymbol{x}=\lambda\boldsymbol{x}$ by $(1/\lambda)\boldsymbol{A}^{-1}$ to obtain $\boldsymbol{A}^{-1}\boldsymbol{x}=(1/\lambda)\boldsymbol{x}$, q.e.d.

(b) If $\lambda=0$ is an eigenvalue the matrix $\boldsymbol{A}$ is singular since $\boldsymbol{A}\boldsymbol{x}=0\cdot\boldsymbol{x}=\boldsymbol{0}$ with $\boldsymbol{x}\neq\boldsymbol{0}$ thereby stating that columns of $\boldsymbol{A}$ are linearly dependent.

(25) (a) Assume $\boldsymbol{A}$ admits an eigendecomposition, $\boldsymbol{A}=\boldsymbol{X}\boldsymbol{\Lambda}\boldsymbol{X}^{-1}$, then

$$\det(\boldsymbol{A})=\det(\boldsymbol{X})\det(\boldsymbol{\Lambda})\det(\boldsymbol{X}^{-1})=\prod_{j=1}^{m}\lambda_j>1$$

since $\det(\boldsymbol{X})^{-1} = \det(\boldsymbol{X}^{-1})$. Since the product is greater than one, at least one factor must be greater than one.

(EC4.b. 1 point) Prove the above when $\boldsymbol{A}$ is defective (automatically awarded is this case was considered in the submitted homework proof).

(b) No. Counter example

$$\boldsymbol{A} = \begin{pmatrix} 8 & 0 \\ 0 & \frac{1}{16} \end{pmatrix}$$

(26) See (24b)

5. (Computer application 4 course points) We illustrate practical applications of eigenvalue and eigenvector computation by an investigation of systems of masses and springs.

Task 1. (1 course point *ex oficio*). Read Section 6.1 of textbook, pp. 293-301.

Task 2. (1 course point). Consider a one-dimensional lattice of n point masses with mass m_j at position j, $1 \leqslant j, k \leqslant n$, connected by $n-1$ springs of stiffness c_j and undeformed length $l = 1$ between point masses $j, j+1$. Let $\boldsymbol{u}$ denote the vector of displacements of the point masseses from their equilibrium positions. Write an Octave/Matlab script to construct the stiffness matrix $\boldsymbol{K}$ of the system (cf. formula 6.12, p.296 of textbook).

Solution. Define a vector of one values, and use the `diag` function to build the stiffness matrix. A more complicated script using loops to individually set each element is possible, but this is the most efficient procedure. Notations follow the textbook. Note the concise construction of $\boldsymbol{A} \in \mathbb{R}^{(n+1)\times n}$ by adding a row of zeros.

```
octave> function K=stiff(C)
          n=max(size(C))-1;
          o=ones(n,1); z= zeros(1,n);
          A=[diag(o); z] - [diag(o(1:n-1),-1); z];
          K=A'*C*A;
        end;
octave>
```

Task 3. (2 course points). Choose $n = \text{ord}(\text{FirstName}) + \text{ord}(\text{LastName})$, with ord the ordinal of the letter in the alphabet. Compute the eigenvalues and eigenvectors of $\boldsymbol{K}$ for:

a) $c_j = 1, 1 \leqslant j \leqslant n-1$

b) $c_j = 1 + (j-1)(n-1-j)$

In both cases, plot the first 5 eigenvectors. What aspect of the motion of the mass-spring system is represented by each eigenvector.

Solution. (a)

```
octave> n=129; C=eye(n); K=stiff(C); disp(K(1:4,1:4));

     2  -1   0   0
    -1   2  -1   0
     0  -1   2  -1
     0   0  -1   2

octave> [X,L]=eig(K);
octave> l=1:n-1;
octave> xlabel('point'); ylabel('displacement'); title('First 5 eigenmodes');
octave> plot(l,X(:,1),l,X(:,2),l,X(:,3),l,X(:,4),l,X(:,5));
octave> cd /home/student/courses/MATH547/homework; print -dpng eigenmodes.png;
octave>
```

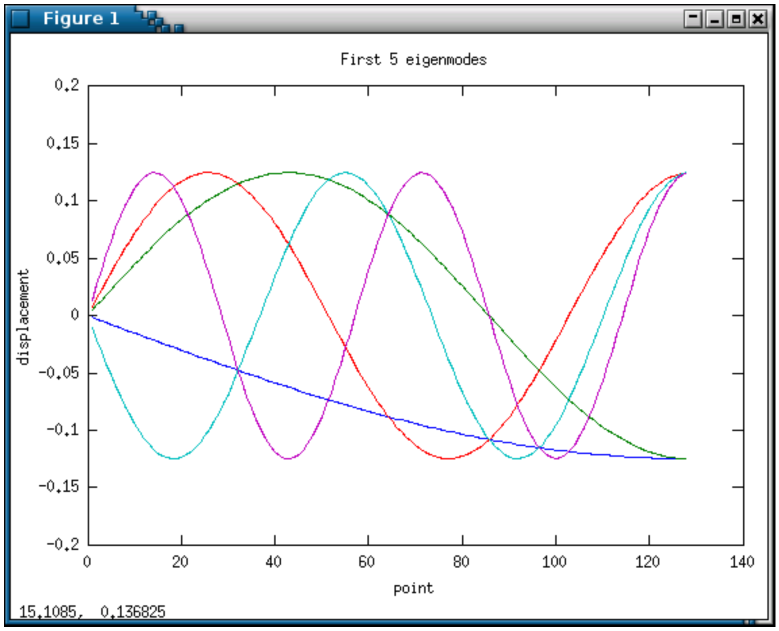

Figure 1. First 5 eigenmodes for $\boldsymbol{C} = \boldsymbol{I}$

(b)

```
octave> n=129;
        for j=1:n
          c(j)=1+(j-1)*(n-1-j);
        end;
octave> C=diag(c); K=stiff(C); [X,L]=eig(K);
octave> l=1:n-1;
octave> xlabel('point'); ylabel('displacement'); title('First 5 eigenmodes');
octave> plot(l,X(:,1),l,X(:,2),l,X(:,3),l,X(:,4),l,X(:,5));
octave> cd /home/student/courses/MATH547/homework; print -dpng eigenmodes.png;
octave>
```

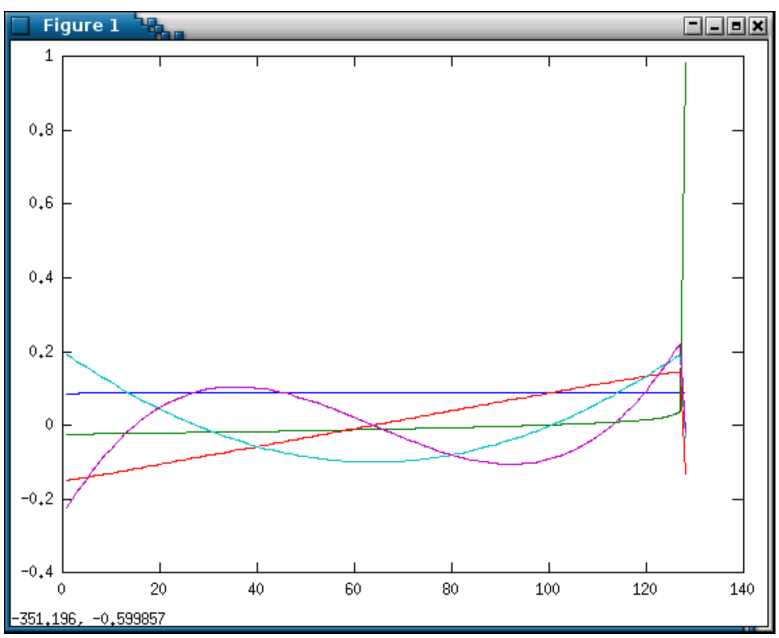

Figure 2. Eigenmodes for stiff-at-center springs