Homework 6 Solution

Due date: April 11, 2016, 11:55PM.
Bibliography: Course lecture notes Lessons 21-26. Textbook pp. 395-424, Sections 8.2-8.4.

We continue study of the properties and applications of eigenvalues and eigenvectors. After understanding the basic computational procedures in Q2, Q3, feel free to use Octave to carry out arithmetic.

1. (1 course point) Textbook p.409, Exercises 8.3.11-8.3.14

8.3.11. By definition $\boldsymbol{A}$ complete admits diagonalization $\boldsymbol{A}=\boldsymbol{X}\boldsymbol{D}\boldsymbol{X}^{-1}$ (i.e., eigenvector matrix $\boldsymbol{X}$ is invertible), hence $\boldsymbol{B}=\boldsymbol{S}(\boldsymbol{X}\boldsymbol{D}\boldsymbol{X}^{-1})\boldsymbol{S}^{-1}=\boldsymbol{T}\boldsymbol{D}\boldsymbol{T}^{-1}$, with $\boldsymbol{T}=\boldsymbol{S}\boldsymbol{X}$, invertible $\boldsymbol{T}^{-1}=(\boldsymbol{S}\boldsymbol{X})^{-1}=\boldsymbol{X}^{-1}\boldsymbol{S}^{-1}$ since $\boldsymbol{S},\boldsymbol{X}$ are invertible.

8.3.12. ($\boldsymbol{U}$ diagonal with equal elements on diagonal $\Rightarrow$ $\boldsymbol{U}$ complete). Proof: By direct construction

$$\boldsymbol{U}=\boldsymbol{I}\boldsymbol{D}\boldsymbol{I}^{-1}, \boldsymbol{D}=\operatorname{diag}(a,a,\ldots,a)$$

($\boldsymbol{U}$ complete $\Rightarrow$ $\boldsymbol{U}$ diagonal). By definition of completeness, and properties of $\boldsymbol{U}$ there exists $\boldsymbol{X}$ invertible such that

$$\boldsymbol{U}=\begin{pmatrix} a & u_{21} & \ldots & u_{m1} \\ 0 & a & \ldots & u_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & a \end{pmatrix}=\boldsymbol{X}\begin{pmatrix} a & 0 & \ldots & 0 \\ 0 & a & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & a \end{pmatrix}\boldsymbol{X}^{-1}\Rightarrow\boldsymbol{U}\boldsymbol{X}=\boldsymbol{X}\boldsymbol{D}$$

Rewrite as

$$\boldsymbol{B}=\begin{pmatrix} a & u_{21} & \ldots & u_{m1} \\ 0 & a & \ldots & u_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & a \end{pmatrix}\begin{pmatrix} \boldsymbol{x}_1 & \boldsymbol{x}_2 & \ldots & \boldsymbol{x}_m \end{pmatrix}=\begin{pmatrix} \boldsymbol{x}_1 & \boldsymbol{x}_2 & \ldots & \boldsymbol{x}_m \end{pmatrix}\begin{pmatrix} a & 0 & \ldots & 0 \\ 0 & a & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & a \end{pmatrix}$$

Identify columns of lhs, rhs

(Column 1)

$$a\boldsymbol{x}_1+u_{21}\boldsymbol{x}_2+\ldots+u_{m1}\boldsymbol{x}_m=a\boldsymbol{x}_1.$$

Since $\boldsymbol{X}$ is invertible, columns of $\boldsymbol{X}$ are linearly independent hence from above, $u_{21}=\ldots=u_{m1}=0$. Similar for remaining columns.

2. (1 course point) Textbook p.412, Exercise 8.3.15
3. (1 course point) Textbook p.412, Exercise 8.3.20
4. (1 course point) Textbook p.412, Exercises 8.3.23, 8.3.28

8.3.28. ($\boldsymbol{A}$, $\boldsymbol{B}$ complete with same eigenvalues with multiplicities $\Rightarrow$ $\boldsymbol{A}\sim\boldsymbol{B}$). By definition of completeness $\exists\boldsymbol{X},\boldsymbol{Y}$ nonsingular eigenvector matrices for $\boldsymbol{A},\boldsymbol{B}$ such that

$$\boldsymbol{A}=\boldsymbol{X}\boldsymbol{D}\boldsymbol{X}^{-1}, \boldsymbol{B}=\boldsymbol{Y}\boldsymbol{D}\boldsymbol{Y}^{-1}\Rightarrow\boldsymbol{D}=\boldsymbol{Y}^{-1}\boldsymbol{B}\boldsymbol{Y}\Rightarrow$$

$$\boldsymbol{A}=\boldsymbol{X}\,(\boldsymbol{Y}^{-1}\boldsymbol{B}\boldsymbol{Y})\boldsymbol{X}^{-1}=\boldsymbol{S}\boldsymbol{B}\boldsymbol{S}^{-1}$$

with $\boldsymbol{S}=\boldsymbol{X}\boldsymbol{Y}^{-1}$, hence $\boldsymbol{A}\sim\boldsymbol{B}$.

($\boldsymbol{A}\sim\boldsymbol{B}$ and both complete $\Rightarrow$ $\boldsymbol{A},\boldsymbol{B}$ have same eigenvalues with multiplicities). $\boldsymbol{A}\sim\boldsymbol{B}\Rightarrow\exists\boldsymbol{S}$ such that $\boldsymbol{B}=\boldsymbol{S}^{-1}\boldsymbol{A}\boldsymbol{S}$. Since $\boldsymbol{A},\boldsymbol{B}$ are complete $\exists\boldsymbol{X},\boldsymbol{Y}$ nonsingular such that $\boldsymbol{A}=\boldsymbol{X}\boldsymbol{D}\boldsymbol{X}^{-1}$ and $\boldsymbol{B}=\boldsymbol{Y}\boldsymbol{E}\boldsymbol{Y}^{-1}$. From eigenvalue relation $\boldsymbol{B}\boldsymbol{y}=\mu\boldsymbol{y}$, obtain $\boldsymbol{A}\,(\boldsymbol{S}\boldsymbol{y})=\mu\,(\boldsymbol{S}\boldsymbol{y})$, or that $\boldsymbol{A}\boldsymbol{x}=\mu\boldsymbol{x}$, with eigenvector $\boldsymbol{x}=\boldsymbol{S}\boldsymbol{y}$ of matrix $\boldsymbol{A}$. Repeating for all eigenvectors gives $\boldsymbol{X}=\boldsymbol{S}\boldsymbol{Y}$, or $\boldsymbol{S}=\boldsymbol{X}\boldsymbol{Y}^{-1}$. Obtain

$$\boldsymbol{B}=\boldsymbol{Y}\boldsymbol{E}\boldsymbol{Y}^{-1}=\boldsymbol{Y}\boldsymbol{X}^{-1}(\boldsymbol{X}\boldsymbol{D}\boldsymbol{X}^{-1})\boldsymbol{X}\boldsymbol{Y}^{-1}=\boldsymbol{Y}\boldsymbol{D}\boldsymbol{Y}^{-1}\Rightarrow\boldsymbol{E}=\boldsymbol{D}.$$

5. (Computer application 4 course points) We continue our investigation of systems of masses and springs.

Task 1. (2 course points). Read Section 9.5 of textbook, pp. 485-491. Write an Octave/Matlab script to reproduce Fig. 9.7 and 9.8 from textbook, and show the motion of the 3 point masses from Example 9.36 (p. 491).

Task 2. (1 course point). Write Octave code to reproduce Fig. 9.10, p.498 of textbook.

Task 3. (1 course point). Write Octave code to reproduce Fig. 9.11, Fig. 9.12, pp.502-503 of textbook.

Hint. In the above you're asked to generate plots of time-varying functions. Here's the procedure to reproduce Fig. 9.5, p. 487.

Theory $u(t)=v\cos(\omega t)$, $\omega=2\pi/T=2\pi f$. Set parameters $T=1$, $v=1$.

```
octave> T=1.; v=1.; omega=2*pi/T;
octave>
```

Choose a time interval containing two periods, and sample it with step size $\Delta t=T/50$.

```
octave> dt=T/50.; t=0.:dt:2*T;
octave>
```

Evaluate and plot the displacement function $u(t)$

```
octave> u=v*cos(omega*t);
octave> plot(t,u,'-b');
octave> xlabel('t'); ylabel('u'); title('Displacement as a function of time');
octave> cd /home/student/courses/MATH547/homework; print -mono -deps hw5Fig9.5.eps;
octave>
```

The above instructions will produce a plot somewhere on the screen. Using a screen capture utility, save the displayed figure to a png file and import it into your writeup

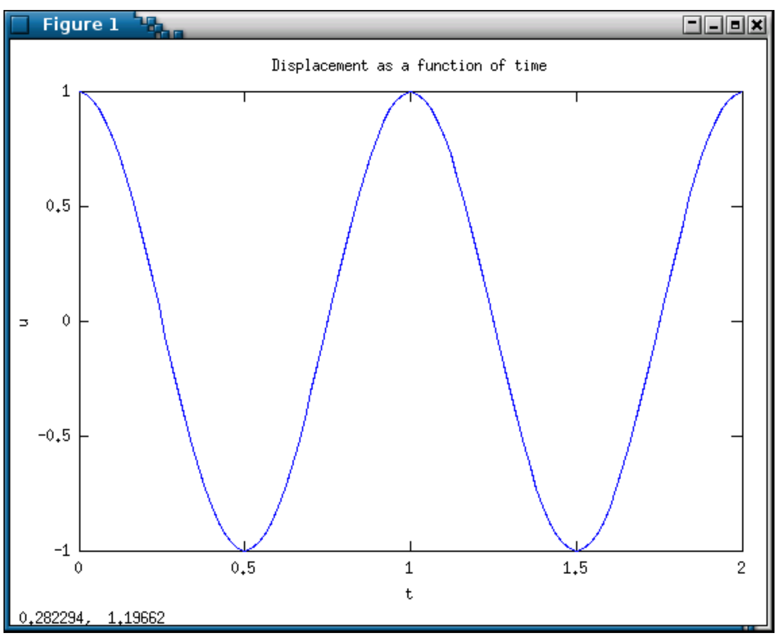

Figure 1. Reproduction of Fig. 9.5 of textbook.

(Fig. 9.10) By numerical experimentation with values for A, μ, ν obtain the plot below.

```
octave> T=1.; A=1.; mu=0.4; nu=4.; del=0.;
octave> dt=T/50.; t=0.:dt:6*T;
octave> u=A*exp(-mu*t).*cos(nu*t);
```

```
octave> plot(t,u,'-b'); xlabel('t'); ylabel('u');
octave> title('Underdamped');
octave> cd /home/student/courses/MATH547/homework; print -mono -deps hw5Fig9.10.eps;
octave>
```