HOMEWORK 7 & 8

Due date: December 2, 2015, 11:55PM. Since multiple submissions are allowed in Sakai, submit after completing some part of the homework to avoid last minute time crunch, and/or computer failure problems.

The final homework of the course helps you prepare for the final examination. There are two parts:

1. *Sample problems and solutions*. You are asked to carefully read the following solutions to problems representative of those you can expect on the final. (honor system grading applied, 8 course grade points awarded *ex officio*).
2. *Challenge, course capstone questions*. Six additional questions are presented without solution. These are synthesis questions, covering multiple aspects of the course, and you should use them to guide review of course concepts.

1 Final examination preparation

1. Consider the quadrilateral formed by points

$$A_0=\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}, A_1=\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}=\begin{pmatrix} 3 \\ -2 \end{pmatrix}, A_2=\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}=\begin{pmatrix} 2 \\ 5 \end{pmatrix}, A_3=\begin{pmatrix} x_3 \\ y_3 \end{pmatrix}=\begin{pmatrix} 5 \\ 3 \end{pmatrix}.$$

   a) Do these points form a parallelogram?

   b) What is the area of the quadrilateral?

   Solution. (a) The quadrilateral has edges defined by vectors

$$B_1=A_1-A_0=\begin{pmatrix} 3 \\ -2 \end{pmatrix}, B_2=A_2-A_0=\begin{pmatrix} 2 \\ 5 \end{pmatrix}, B_3=A_3-A_1=\begin{pmatrix} 2 \\ 5 \end{pmatrix}, B_4=A_3-A_2=\begin{pmatrix} 3 \\ -2 \end{pmatrix}.$$

   Since $B_1=B_4$, $B_2=B_3$, the quadrilateral is indeed a parallelogram. (b) The area of a parallelogram is given by the determinant formed by edges

$$\text{Area}=\det([B_1\, B_2])=\begin{vmatrix} 3 & 2 \\ -2 & 5 \end{vmatrix}=19.$$

2. Compute the value of the determinant

$$\Delta=\begin{vmatrix} 1 & 2 & -1 & -1 \\ -3 & 0 & 2 & 1 \\ 2 & -1 & 5 & 4 \\ -1 & 6 & 3 & 3 \end{vmatrix}.$$

   Solution. The strategy is to produce zeros in a row and column and then expand in algebraic minors. The value of the determinant is preserved by linear combination of rows or columns. Carry out linear combinations of column 1 with columns 2,3,4

$$\Delta=\begin{vmatrix} 1 & 0 & 0 & 0 \\ -3 & 6 & -1 & -2 \\ 2 & -5 & 7 & 6 \\ -1 & 8 & 2 & 2 \end{vmatrix}$$

   and expand along first row

$$\Delta=\begin{vmatrix} 6 & -1 & -2 \\ -5 & 7 & 6 \\ 8 & 2 & 2 \end{vmatrix}.$$

Now carry out linear combinations of column 2 with columns 1,3

$$\Delta = \begin{vmatrix} 0 & -1 & 0 \\ 37 & 7 & -8 \\ 20 & 2 & -2 \end{vmatrix}$$

Expand along first row

$$\Delta = \begin{vmatrix} 37 & -8 \\ 20 & -2 \end{vmatrix} = -74 + 160 = 86.$$

3. Solve the following linear system by Cramer's rule

$$\begin{cases} 2x_1 + x_2 - x_3 = 1 \\ x_1 - 2x_2 + x_3 = 0 \\ 3x_1 + 4x_2 - 2x_3 = -5 \end{cases}$$

Solution. The principal determinant of the system is

$$\Delta = \begin{vmatrix} 2 & 1 & -1 \\ 1 & -2 & 1 \\ 3 & 4 & -2 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 0 \\ 5 & -2 & -1 \\ -5 & 4 & 2 \end{vmatrix} = -\begin{vmatrix} 5 & -1 \\ -5 & 2 \end{vmatrix} = -5$$

Replacing column i in Δ by the rhs term for $i = 1, 2, 3$ gives

$$x_1 = \frac{\Delta_1}{\Delta} = -\frac{1}{5}\begin{vmatrix} 1 & 1 & -1 \\ 0 & -2 & 1 \\ -5 & 4 & -2 \end{vmatrix} = -\frac{1}{5}\begin{vmatrix} 1 & 0 & 0 \\ 0 & -2 & 1 \\ -5 & 9 & -7 \end{vmatrix} = -\frac{1}{5}\begin{vmatrix} -2 & 1 \\ 9 & -7 \end{vmatrix} = -1$$

$$x_2 = \frac{\Delta_2}{\Delta} = -\frac{1}{5}\begin{vmatrix} 2 & 1 & -1 \\ 1 & 0 & 1 \\ 3 & -5 & -2 \end{vmatrix} = -\frac{1}{5}\begin{vmatrix} 0 & 0 & -1 \\ 3 & 1 & 1 \\ -1 & -7 & -2 \end{vmatrix} = \frac{1}{5}\begin{vmatrix} 3 & 1 \\ -1 & -7 \end{vmatrix} = -4$$

$$x_3 = \frac{\Delta_3}{\Delta} = -\frac{1}{5}\begin{vmatrix} 2 & 1 & 1 \\ 1 & -2 & 0 \\ 3 & 4 & -5 \end{vmatrix} = -\frac{1}{5}\begin{vmatrix} 0 & 1 & 0 \\ 5 & -2 & 2 \\ -5 & 4 & -8 \end{vmatrix} = \frac{1}{5}\begin{vmatrix} 5 & 2 \\ -5 & -8 \end{vmatrix} = -7$$

4. What is the volume of the parallelepiped with edges

$$A_1 = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}, A_2 = \begin{pmatrix} -1 \\ 3 \\ -5 \end{pmatrix}, A_3 = A_1 \times A_2?$$

Solution. The A_3 edge components are computed from the cross product

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 4 \\ -1 & 3 & -5 \end{vmatrix} = -22\vec{i} + \vec{j} + 5\vec{k}.$$

The parallelepiped volume is given by the determinant

$$V = \begin{vmatrix} 1 & 2 & 4 \\ -1 & 3 & -5 \\ -22 & 1 & 5 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ -1 & 5 & -1 \\ -22 & 45 & 93 \end{vmatrix} = 5\begin{vmatrix} 1 & -1 \\ 9 & 93 \end{vmatrix} = 510.$$

5. Find the eigenvalues and eigenvectors of the matrix $A = uu^T$ with $u^T = (\begin{matrix} 1 & 2 & 1 \end{matrix})$.

Solution. The matrix

$$A = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{pmatrix}$$

is of dimensions 3×3 and of rank 1. We expect to find only one non-zero eigenvalue. The characteristic determinant is

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 2 & 1 \\ 2 & 4-\lambda & 2 \\ 1 & 2 & 1-\lambda \end{vmatrix} = \begin{vmatrix} 1-\lambda & 2\lambda & \lambda \\ 2 & -\lambda & 0 \\ 1 & 0 & -\lambda \end{vmatrix} = \lambda^2 \begin{vmatrix} 1-\lambda & 2 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & -1 \end{vmatrix} =$$

$$= \lambda^2 \begin{vmatrix} 1-\lambda & 2 & 2-\lambda \\ 2 & -1 & 2 \\ 1 & 0 & 0 \end{vmatrix} = \lambda^2 \begin{vmatrix} 2 & 2-\lambda \\ -1 & 2 \end{vmatrix} = \lambda^2(6-\lambda) \Rightarrow \lambda_1 = \lambda_2 = 0, \lambda_3 = 6.$$

Since $\operatorname{rank}(A) = 1$ we know that by row echelon reduction

$$A \sim \begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and the two eigenvectors associated with repeated eigenvalue $\lambda_{1,2} = 1$ are

$$x_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, x_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

(basis for $\operatorname{null}(A)$). For $\lambda_3 = 6$, row echelon reduction leads to

$$A - 6I = \begin{pmatrix} -5 & 2 & 1 \\ 2 & -2 & 2 \\ 1 & 2 & -5 \end{pmatrix} \sim \begin{pmatrix} 0 & 0 & 0 \\ 0 & -6 & 12 \\ 1 & 2 & -5 \end{pmatrix} \Rightarrow x_3 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

6. A real-valued matrix $A \in \mathbb{R}^{m \times m}$ is skew-symmetric if $A = -A^T$. Prove that the eigenvalues of a skew-symmetric matrix are purely imaginary (i.e. complex numbers with zero real part).

   Solution. The complex conjugate of the eigenvalue relationship $Ax = \lambda x$ is $A\bar{x} = \bar{\lambda}\bar{x}$ with transpose

$$\bar{x}^T A^T = \bar{\lambda} \bar{x}^T \Rightarrow -\bar{x}^T A = \bar{\lambda} \bar{x}^T.$$

   Multiply eigenvalue relation on left by $\bar{x}^T$, above relation on left by x to obtain

$$\bar{x}^T A x = \lambda \bar{x}^T x, -\bar{x}^T A x = \bar{\lambda} \bar{x}^T x \Rightarrow -\lambda = \bar{\lambda}$$

   since $x \neq 0$. The real part of a complex number is given by $\operatorname{Re} \lambda = (\lambda + \bar{\lambda})/2 = 0$, so λ is purely imaginary.

7. Prove that

$$A^k = \frac{1}{2} \begin{pmatrix} 1+3^k & 1-3^k \\ 1-3^k & 1+3^k \end{pmatrix}$$

   for

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

   Solution. The matrix A is symmetric, hence diagonalizable, $A = Q \Lambda Q^T$, with Q an orthogonal matrix, such that

$$A^k = (Q\Lambda Q^T)^k = (Q\Lambda Q^T)(Q\Lambda Q^T)...(Q\Lambda Q^T) = Q\Lambda^k Q^T$$

The characteristic polynomial of A is

$$\det(A-\lambda I) = \begin{vmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} = 3-4\lambda+\lambda^2$$

with roots $\lambda_1 = 1$, $\lambda_2 = 3$. From

$$A-\lambda_1 I = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$$

obtain unit-norm eigenvector

$$x_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

and from

$$A-\lambda_2 I = \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

obtain unit-norm eigenvector

$$x_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

The eigenvector matrix is

$$Q = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, Q^T = Q^{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Carry out the computation

$$Q\Lambda^k Q^T = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 3^k \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 3^k & -3^k \end{pmatrix}$$

$$Q\Lambda^k Q^T = \frac{1}{2}\begin{pmatrix} 1+3^k & 1-3^k \\ 1-3^k & 1+3^k \end{pmatrix}.$$

8. Compute the singular value decomposition of $A = ww^T$ with $w^T = (\,1\;\;2\;\;1\,)$.

Solution. The singular value decomposition of $A \in \mathbb{R}^{3\times 3}$ is $A = U\Sigma V^T$ with $U, V \in \mathbb{R}^{3\times 3}$ orthogonal matrices and $\Sigma = \operatorname{diag}(\sigma_1, \sigma_2, \sigma_3)$. For symmetric A, we have $A = U\Sigma U^T$. The SVD can be expressed as a sum of rank-1 contributions

$$A = (\,U_1\;\;U_2\;\;U_3\,)\begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix}\begin{pmatrix} U_1^T \\ U_2^T \\ U_3^T \end{pmatrix} = \sigma_1 U_1 U_1^T + \sigma_2 U_2 U_2^T + \sigma_3 U_3 U_3^T.$$

For $A = ww^T$ there is a single rank-1 term hence $\sigma_2 = \sigma_3 = 0$. From $\|w\| = \sqrt{6}$, we can write

$$A = 6U_1 U_1^T, U_1 = \frac{1}{\sqrt{6}}w = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}.$$

To complete U, choose U_2 as a unit vector orthogonal to U_1

$$U_2 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

and $U_3 = U_1 \times U_2$ (by properties of cross product U_3 will be of unit norm and orthogonal to both U_1 and U_2)

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 1 \\ 1 & -1 & 1 \end{vmatrix} = 3\vec{i} - 3\vec{k} \Rightarrow U_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

The SVD is

$$A=\begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{pmatrix}=\begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 6 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

2 Additional course capstone questions

1. Determine the singular value decomposition of

$$A=\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

2. Determine s such that

$$A=\begin{pmatrix} s & -4 & -4 \\ -4 & s & -4 \\ -4 & -4 & s \end{pmatrix}$$

is positive definite.

3. Find a matrix $A \neq 0$ for which $A^3 = 0$. What are the eigenvalues of A?
4. If $B \in \mathbb{R}^{3\times 3}$ has eigenvalues 0,1,2 give values (or state that there is not enough information to specify a value) for:
   a) $\operatorname{rank}(B)$
   b) $\det(B^T B) = |B^T B|$
   c) eigenvalues of $B^T B$
   d) eigenvalues of $(B^2 + I)^{-1}$
5. Write the rotation matrix R_θ of angle θ around axis $u \in \mathbb{R}^3$, $\|u\| = 1$. Find the eigenvalues and eigenvectors of R_θ.
6. Derive formulas for the inverse and determinant of Hadamard matrices of order m, matrices with orthogonal rows/columns and entries equal to either 1 or -1.