Lesson concepts:

- Vectors
- Vector operations
- Linear combinations
- Matrix vector multiplication

- Some quantities arising in applications can be expressed as single numbers, called "scalars"
    - Speed of a car on a highway $v = 35$ mph
    - A person's height $H = 183$ cm
- Many other quantitites require more than one number:
    - Position in a city: "Intersection of 86th St and 3rd Av"
    - Position in 3D space: (x, y, z)
    - Velocity in 3D space: (u, v, w)

Definition. *A* *vector* *is a grouping of* m *scalars*

$$\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{pmatrix} \in S^m, v_i \in S$$

- The scalars usually are naturals ($S = \mathbb{N}$), integers ($S = \mathbb{Z}$), rationals ($S = \mathbb{Q}$), reals ($S = \mathbb{R}$), or complex numbers ($S = \mathbb{C}$)
- We often denote the dimension and set of scalars through the notation $\boldsymbol{v} \in S^m$, e.g. $\boldsymbol{v} \in \mathbb{R}^m$
- Sets of vectors are denoted as

$$\mathcal{V} = \{\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{pmatrix}, v_i \in S\} \tag{1}$$

- A vector can also be interpreted as a function from a subset of $\mathbb{N}$ to S

$$v: \{1, 2, .., m\} \rightarrow S$$

- **Vector addition**. Consider two vectors $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{V}$. We define the sum of the two vectors as the vector containing the sum of the components

$$\boldsymbol{w} = \boldsymbol{u} + \boldsymbol{v} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{pmatrix} = \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \vdots \\ u_m + v_m \end{pmatrix} = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_m \end{pmatrix}$$

- **Scalar multiplication**. Consider $\alpha \in S$, $\boldsymbol{u} \in \mathcal{V}$. We define the multiplication of vector $\boldsymbol{u}$ by scalar α as the vector containing the product of each component of $\boldsymbol{u}$ with the scalar α

$$\boldsymbol{w} = \alpha\ \boldsymbol{u} = \begin{pmatrix} \alpha\ u_1 \\ \alpha\ u_2 \\ \vdots \\ \alpha\, u_m \end{pmatrix} = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_m \end{pmatrix}$$

- **Linear combination**. Let $\alpha, \beta \in S$, $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{V}$. Define a linear combination of vectors by

$$\boldsymbol{w} = \alpha\ \boldsymbol{u} + \beta\ \boldsymbol{v} = \begin{pmatrix} \alpha\, u_1 \\ \alpha\ u_2 \\ \vdots \\ \alpha\ u_m \end{pmatrix} + \begin{pmatrix} \beta\ v_1 \\ \beta\ v_2 \\ \vdots \\ b\beta\, v_m \end{pmatrix} = \begin{pmatrix} \alpha\ u_1 + \beta\ v_1 \\ \alpha\ u_2 + \beta\ v_2 \\ \vdots \\ \alpha\, u_m + \beta\, v_n \end{pmatrix} = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_m \end{pmatrix}$$

"Start at the center of town. Go east 3 blocks and north 2 blocks. What is your final position?"

$$\boldsymbol{s} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \boldsymbol{e}_E = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \boldsymbol{e}_N = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\alpha_E = 3, \alpha_N = 2$$

$$\boldsymbol{f} = \boldsymbol{s} + \alpha_E \boldsymbol{e}_E + \alpha_N \boldsymbol{e}_N = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + 3 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

Linear combinations allow us to express a position in space using a standard set of directions.
Questions:

- How many standard directions are needed?
- Can any position be specified as a linear combination?
- How to find the scalars needed to express a position as a linear combination?

Seek a more compact notation for the linear combination

$$a\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + b\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + c\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} a-b+2c \\ 2a \\ 3a+b+c \end{pmatrix}$$

- Group the vectors together to form a "matrix"

$$\boldsymbol{A} = \begin{pmatrix} 1 & -1 & 2 \\ 2 & 0 & 0 \\ 3 & 1 & 1 \end{pmatrix}$$

- Group the scalars together to form a vector

$$\boldsymbol{u} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

- Define matrix-vector multiplication

$$\boldsymbol{A}\,\boldsymbol{u} = \begin{pmatrix} 1 & -1 & 2 \\ 2 & 0 & 0 \\ 3 & 1 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a-b+2c \\ 2a \\ 3a+b+c \end{pmatrix}$$