New concepts:

- Matrix-matrix product
- Matrix transpose
- Transpose of matrix sums, products
- Scalar product of two vectors
Definition. Consider matrices \( \mathbf{A} = ( \mathbf{a}_1 \ldots \mathbf{a}_n ) \in \mathbb{R}^{m \times n} \), and \( \mathbf{X} = ( \mathbf{x}_1 \ldots \mathbf{x}_p ) \in \mathbb{R}^{n \times p} \). The matrix product \( \mathbf{B} = \mathbf{A} \mathbf{X} \) is a matrix \( \mathbf{B} = ( \mathbf{b}_1 \ldots \mathbf{b}_p ) \in \mathbb{R}^{m \times p} \) with column vectors given by the matrix vector products

\[
\mathbf{b}_k = \mathbf{A} \mathbf{x}_k, \text{ for } k = 1, 2, \ldots, p.
\]

- A matrix-matrix product is simply a set of matrix-vector products, and hence expresses multiple linear combinations in a concise way.
- The dimensions of the matrices must be compatible, the number of rows of \( \mathbf{X} \) must equal the number of columns of \( \mathbf{A} \).
- A matrix-vector product is a special case of a matrix-matrix product when \( p = 1 \).
- We often write \( \mathbf{B} = \mathbf{A} \mathbf{X} \) in terms of columns as

\[
( \mathbf{b}_1 \ldots \mathbf{b}_p ) = \mathbf{A} ( \mathbf{x}_1 \ldots \mathbf{x}_p ) = ( \mathbf{A} \mathbf{x}_1 \ldots \mathbf{A} \mathbf{x}_p )
\]
**Definition.** Given a matrix \( A \in \mathbb{R}^{m \times n} \),

\[
A = \begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{pmatrix}
\]

the **transpose** of \( A \), denoted as \( A^T \in \mathbb{R}^{n \times m} \) is

\[
A^T = \begin{pmatrix}
a_{11} & a_{21} & \cdots & a_{n1} \\
a_{12} & a_{22} & \cdots & a_{n2} \\
\vdots & \vdots & \ddots & \vdots \\
a_{1m} & a_{2m} & \cdots & a_{nm}
\end{pmatrix}
\]

Intuitively: “rows become columns, columns become rows”
• Recall that a vector is the special case of a matrix with a single column, \( \mathbf{v} \in \mathbb{R}^{m\times 1} \). The transpose of a vector is \( \mathbf{v}^T \in \mathbb{R}^{1\times m} \) a matrix with a single row, known as a \textit{row vector}.

• Given a matrix \( \mathbf{A} \in \mathbb{R}^{m\times n} \) expressed through its column vectors

\[
\mathbf{A} = (\begin{array}{cccc}
    a_1 & a_2 & \ldots & a_n
  \end{array}),
\]

its transpose can be expressed as

\[
\mathbf{A}^T = \begin{pmatrix}
    a_1^T \\
    a_2^T \\
    \vdots \\
    a_n^T
\end{pmatrix} \in \mathbb{R}^{n\times m}.
\]
- For $\mathbf{u}, \mathbf{v} \in \mathbb{R}^m$, $(\mathbf{u} + \mathbf{v})^T = \mathbf{u}^T + \mathbf{v}^T$. Proof: by direct computation

$$(\mathbf{u} + \mathbf{v})^T = \begin{pmatrix}
\begin{pmatrix}
\mathbf{u}_1 \\
\vdots \\
\mathbf{u}_m
\end{pmatrix} + 
\begin{pmatrix}
\mathbf{v}_1 \\
\vdots \\
\mathbf{v}_m
\end{pmatrix}
\end{pmatrix}^T = 
\begin{pmatrix}
\mathbf{u}_1 + \mathbf{v}_1 \\
\vdots \\
\mathbf{u}_m + \mathbf{v}_m
\end{pmatrix}^T = 
\begin{pmatrix}
\mathbf{u}_1 + \mathbf{v}_1 & \ldots & \mathbf{u}_m + \mathbf{v}_m
\end{pmatrix}
= (\begin{pmatrix}
\mathbf{u}_1 \\
\vdots \\
\mathbf{u}_m
\end{pmatrix}) + (\begin{pmatrix}
\mathbf{v}_1 \\
\vdots \\
\mathbf{v}_m
\end{pmatrix}) = \mathbf{u}^T + \mathbf{v}^T$$

- For $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{m \times n}$, $(\mathbf{A} + \mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T$. Proof: by direct computation using column vectors of $\mathbf{A}, \mathbf{B}$

$$\mathbf{A} = (\begin{pmatrix}
\mathbf{a}_1 \\
\vdots \\
\mathbf{a}_n
\end{pmatrix}), \mathbf{B} = (\begin{pmatrix}
\mathbf{b}_1 \\
\vdots \\
\mathbf{b}_n
\end{pmatrix})$$

$$(\mathbf{A} + \mathbf{B})^T = (\begin{pmatrix}
\mathbf{a}_1 + \mathbf{b}_1 \\
\vdots \\
\mathbf{a}_n + \mathbf{b}_n
\end{pmatrix})^T = 
\begin{pmatrix}
(\mathbf{a}_1 + \mathbf{b}_1)^T \\
\vdots \\
(\mathbf{a}_n + \mathbf{b}_n)^T
\end{pmatrix} = 
\begin{pmatrix}
\mathbf{a}_1^T + \mathbf{b}_1^T \\
\vdots \\
\mathbf{a}_n^T + \mathbf{b}_n^T
\end{pmatrix} = \mathbf{A}^T + \mathbf{B}^T$$
Consider $A \in \mathbb{R}^{m \times n}$, $x \in \mathbb{R}^n$. What is $(Ax)^T$? Recall that $Ax$ is a linear combination of columns of $A$

$$Ax = (a_1 \ldots a_n)x = x_1a_1 + \ldots + x_na_n$$

Take transpose of vector sum to obtain

$$(Ax)^T = x_1a_1^T + \ldots + x_na_n^T = (x_1 \ldots x_n)\begin{pmatrix} a_1^T \\ \vdots \\ a_n^T \end{pmatrix} = x^TA^T \quad (1)$$

Now consider $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{n \times p}$. Obtain $(AB)^T = B^TA^T$ by applying (1)

$$(AB)^T = (Ab_1 \ldots Ab_n)^T = ( (Ab_1)^T \\ \vdots \\ (Ab_n)^T ) = ( b_1^TA^T \\ \vdots \\ b_n^TA^T ) = ( b_1^T \\ \vdots \\ b_n^T )A^T = B^TA^T$$
A special case of a matrix-matrix product occurs when the two factors correspond to a row multiplying a column vector. The result is in this case a single scalar.

**Definition.** Given two real-valued (column) vectors \( u, v \in \mathbb{R}^m \) the scalar product is the matrix-matrix multiplication

\[
u^T v = \left( \begin{array}{c} u_1 \\ u_2 \\ \vdots \\ u_m \end{array} \right) \left( \begin{array}{c} v_1 \\ v_2 \\ \vdots \\ v_m \end{array} \right) = \sum_{i=1}^{m} u_i v_i
\]

Note the compatibility of number of components, \( u \in \mathbb{R}^{m \times 1} \Rightarrow u^T \in \mathbb{R}^{1 \times m}, v \in \mathbb{R}^{m \times 1}, u^T v \in \mathbb{R}^{1 \times 1} = \mathbb{R} \).

The scalar product can be viewed as function taking two vectors as arguments and producing a single scalar as a result. The usual notation in this case is

\[
\langle , \rangle : \mathcal{V} \times \mathcal{V} \to \mathbb{R}, \langle u, v \rangle = u^T v = \sum_{i=1}^{m} u_i v_i
\]

with \( \mathcal{V} = \mathbb{R}^m \).