- *Motivation*. We've introduced the basis entities of linear algebra: vectors and matrices. We've also introduced the operation of most interest in linear algebra: linear combination expressed as a matrix vector product. A *measurement* is a comparison of a scalar quantity with respect to some chosen unit. How can the notion of measurement be extended to groupings of scalars, such as vectors and matrices?
- New concepts:
  - Scalar product of two vectors
  - Orthogonal vectors
  - Norm of a vector

Scalar products prove to useful in many other contexts than real-valued vectors.

Definition. *Consider vectors* $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \mathcal{V}$ *and scalar* $a \in S$. *The function*

$$\langle\, ,\rangle : \mathcal{V} \times \mathcal{V} \to S$$

is a scalar product if:

1. $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \overline{\langle \boldsymbol{v}, \boldsymbol{u} \rangle}$ *(Conjugate symmetry)*
2. $\langle a\boldsymbol{u}, \boldsymbol{v} \rangle = a\langle \boldsymbol{u}, \boldsymbol{v} \rangle, \langle \boldsymbol{u} + \boldsymbol{v}, \boldsymbol{w} \rangle = \langle \boldsymbol{u}, \boldsymbol{w} \rangle + \langle \boldsymbol{v}, \boldsymbol{w} \rangle$ *(Linearity in first argument)*
3. $\langle \boldsymbol{u}, \boldsymbol{u} \rangle \geqslant 0, \langle \boldsymbol{u}, \boldsymbol{u} \rangle = 0 \Rightarrow \boldsymbol{u} = \boldsymbol{0}$ *(Positive definiteness)*

This definition is constructed to remain valid for complex scalars $S = \mathbb{C}$. Recall the conjugate of a complex number $z = x + iy$ is $\bar{z} = x - iy$.

1. $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \overline{\langle \boldsymbol{v}, \boldsymbol{u} \rangle}$ (Conjugate symmetry)

   Verify:

   a) $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \boldsymbol{u}^T \boldsymbol{v} = \sum_{i=1}^m u_i v_i$ by definition of scalar product of column vectors

   b) $\overline{\langle \boldsymbol{v}, \boldsymbol{u} \rangle} = \overline{\boldsymbol{v}^T \boldsymbol{u}} = \overline{\sum_{i=1}^m v_i u_i}$ by definition of scalar product of column vectors

   c) $\overline{\sum_{i=1}^m v_i u_i} = \sum_{i=1}^m v_i u_i$ since $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^m$

   d) $\sum_{i=1}^m v_i u_i = \sum_{i=1}^m u_i v_i$ since multiplication of scalar is commutative

   e) (a)=(d)$\Rightarrow \langle \boldsymbol{u}, \boldsymbol{v} \rangle = \overline{\langle \boldsymbol{v}, \boldsymbol{u} \rangle}$

2. $\langle a\boldsymbol{u}, \boldsymbol{v} \rangle = a\langle \boldsymbol{u}, \boldsymbol{v} \rangle, \langle \boldsymbol{u} + \boldsymbol{v}, \boldsymbol{w} \rangle = \langle \boldsymbol{u}, \boldsymbol{w} \rangle + \langle \boldsymbol{v}, \boldsymbol{w} \rangle$ (Linearity in first argument)

3. $\langle \boldsymbol{u}, \boldsymbol{u} \rangle \geqslant 0, \langle \boldsymbol{u}, \boldsymbol{u} \rangle = 0 \Rightarrow \boldsymbol{u} = \boldsymbol{0}$ (Positive definiteness)

Definition. *Two vectors $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{V}$ whose scalar product is zero are said to be orthogonal to one another.*

- Consider two standard basis vectors $\boldsymbol{e}_i, \boldsymbol{e}_j \in \mathbb{R}^m$. Their scalar product is

$$\boldsymbol{e}_i^T \boldsymbol{e}_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases},$$

and the column vectors of the identity matrix $\boldsymbol{I}_m$ are therefore orthogonal to one another.

The situation where a scalar product result is one for equal indices and zero otherwise arises often and is given a special notation known as *Kronecker delta*

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}.$$

Note that the components of the identity matrix can be expressed as $\boldsymbol{I}_m = (\delta_{ij})$.

Definition. *The norm of a vector is a function that takes a vector argument, returns a positive real number,* $\| \, \| : \mathcal{V} \to \mathbb{R}_+$*, and for* $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{V}$*,* $a \in S$*,satisfies properties:*

1. $\|a\boldsymbol{u}\| = |a| \, \|\boldsymbol{u}\|$
2. $\|\boldsymbol{u} + \boldsymbol{v}\| \leqslant \|\boldsymbol{u}\| + \|\boldsymbol{v}\|$
3. $\|\boldsymbol{u}\| = 0 \Rightarrow \boldsymbol{u} = \boldsymbol{0}$.

- A norm embodies the concept of measurement of the magnitude of a vector
- Different ways of measuring the magnitude of a vector are most appropriate in various applications, resulting in different definitions of a vector norm for $\boldsymbol{u} \in \mathbb{R}^m$:

  1-norm. $\|\boldsymbol{u}\|_1 = \sum_{i=1}^{m} |u_i|$

  2-norm (Euclidean norm). $\|\boldsymbol{u}\|_2 = \left(\sum_{i=1}^{m} u_i^2\right)^{1/2} = (\boldsymbol{u}^T \boldsymbol{u})^{1/2}$

  inf-norm. $\|\boldsymbol{u}\|_\infty = \max_{i \in \{1,2,\dots,m\}} |u_i|$

- Different norms are distinguished by subscripts as above. The most commonly used norm is the Euclidean norm that corresponds to the square root of the scalar product $\boldsymbol{u}^T\boldsymbol{u}$, in which case the subscript is often suppressed to simplify notation $\|\boldsymbol{u}\| = \|\boldsymbol{u}\|_2$