- New concepts:
  - Norm computation examples
  - Angle between two vectors is defined using scalar product and norm
  - Solving a linear system

Definition. *Consider vector* $\boldsymbol{u} = (u_1 \ \ \ldots \ \ u_m)^T \in \mathbb{R}^m$. *The* p*-norm of* $\boldsymbol{u}$ *is a function* $\| \ \|_p$: $\mathbb{R}^m \to \mathbb{R}_+$ *defined by*

$$\|\boldsymbol{u}\|_p = \left(\sum_{i=1}^{m} |u_i|^p \right)^{1/p}.$$

```
octave> u=[1 2 3]'; unrm1=norm(u,1); disp(unrm1);
octave> unrm2=norm(u,2); disp(unrm2);
octave> unrminf=norm(u,inf); disp(unrminf);
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 3
octave>
```
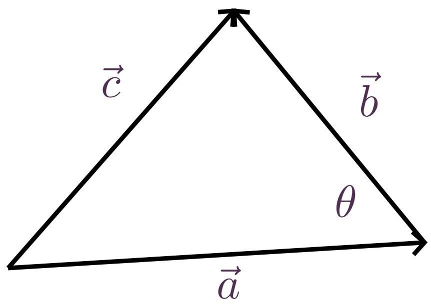

Definition. *Consider vectors* $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^m$. *The angle* θ *between* $\boldsymbol{u}, \boldsymbol{v}$ *is defined by*

$$\cos\theta = \frac{\boldsymbol{u}^T\boldsymbol{v}}{\|\boldsymbol{u}\|\|\boldsymbol{v}\|}.$$

Notes:

1. The angle is defined using the concept of scalar product and norm.
2. Recall that $\boldsymbol{u}, \boldsymbol{v}$ are orthogonal if $\boldsymbol{u}^T\boldsymbol{v} = 0$, which implies $\cos\theta = 0$, hence $\theta = \pi/2$



Cosine theorem: If $\vec{c} = \vec{a} + \vec{b}$ then $c^2 = a^2 + b^2 - 2ab\cos\theta$, with $a = |\vec{a}|, b = |\vec{b}|, c = |\vec{c}|$

Proof using algebraic concepts of norm and scalar product:

$$\begin{aligned} \boldsymbol{c} = \boldsymbol{a} + \boldsymbol{b} \Rightarrow & \quad \boldsymbol{c}^T = \boldsymbol{a}^T + \boldsymbol{b}^T \\ \boldsymbol{c}^T\boldsymbol{c} = (\boldsymbol{a}^T + \boldsymbol{b}^T)(\boldsymbol{a} + \boldsymbol{b}) \Rightarrow & \quad \|\boldsymbol{c}\|^2 = \|\boldsymbol{a}\|^2 + \|\boldsymbol{b}\|^2 + 2\boldsymbol{a}^T\boldsymbol{b} \Rightarrow \\ \|\boldsymbol{c}\|^2 = \|\boldsymbol{a}\|^2 + \|\boldsymbol{b}\|^2 + 2\|\boldsymbol{a}\|\|\boldsymbol{b}\|\cos(\pi - \theta) \Rightarrow & \quad \|\boldsymbol{c}\|^2 = \|\boldsymbol{a}\|^2 + \|\boldsymbol{b}\|^2 - 2\|\boldsymbol{a}\|\|\boldsymbol{b}\|\cos\theta \end{aligned}$$

- We've interpreted $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} = \boldsymbol{I}\boldsymbol{b}$ to signify equality of a two ways of expressing a vector:
  1. As a linear combination of the columns of $\boldsymbol{A}$, namely $\boldsymbol{A}\boldsymbol{x}$
  2. As a linear combination of the columns of $\boldsymbol{I}$, namely $\boldsymbol{I}\boldsymbol{b}$
- One of the basic linear algebra problems is to find the coordinates a vector in terms of columns of $\boldsymbol{A}$ given its coordinates $\boldsymbol{b}$ in terms of the identity matrix $\boldsymbol{I}$
- Example: Find $\boldsymbol{x} \in \mathbb{R}^3$ such that

$$\begin{pmatrix} 1 & 2 & -1 \\ 2 & -1 & 1 \\ 3 & -1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$$

- Equivalent to writing out the system

$$\begin{cases} x_1 + 2\,x_2 - x_3 & = & 2 \\ 2\,x_1 - x_2 + x_3 & = & 2 \\ 3\,x_1 - x_2 - x_3 & = & 1 \end{cases}$$

- Idea: make one fewer unknown appear in each equation. Use first equation to eliminate x_1 in equations 2,3

$$\begin{cases} x_1 + 2\,x_2 - x_3 &= 2 \\ 2\,x_1 - x_2 + x_3 &= 2 \\ 3\,x_1 - x_2 - x_3 &= 1 \end{cases} \Rightarrow \begin{cases} x_1 + 2\,x_2 - x_3 &= 2 \\ -5\,x_2 + 3\,x_3 &= -2 \\ -7\,x_2 + 2\,x_3 &= -5 \end{cases}$$

- Use second equation to eliminate x_2 in equation 3

$$\begin{cases} x_1 + 2\,x_2 - x_3 &= 2 \\ -5\,x_2 + 3\,x_3 &= -2 \\ -7\,x_2 + 2\,x_3 &= -5 \end{cases} \Rightarrow \begin{cases} x_1 + 2\,x_2 - x_3 &= 2 \\ -5\,x_2 + 3\,x_3 &= -2 \\ -\frac{11}{5}\,x_3 &= -\frac{11}{5} \end{cases}$$

- Start finding components from last to first to obtain $x_3 = 1$, $x_2 = 1$,$x_1 = 1$

- Explicitly writing the unknowns x_1, x_2, x_3 is not necessary. Intoduce the "bordered" matrix

$$\begin{pmatrix} 1 & 2 & -1 & 2 \\ 2 & -1 & 1 & 2 \\ 3 & -1 & -1 & 1 \end{pmatrix}$$

- Define allowed operations:
  - multiply a row by a non-zero scalar
  - add a row to another
- Bordered matrices obtained by the allowed operations are said to be *similar*, in that the solution of the linear system stays the same

$$\begin{pmatrix} 1 & 2 & -1 & 2 \\ 2 & -1 & 1 & 2 \\ 3 & -1 & -1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -1 & 2 \\ 0 & -5 & 3 & -2 \\ 0 & -7 & 2 & -5 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -1 & 2 \\ 0 & -5 & 3 & -2 \\ 0 & 0 & -\frac{11}{5} & -\frac{11}{5} \end{pmatrix}$$

- To find solution, use allowed operations to make an identity matrix appear

$$\begin{pmatrix} 1 & 2 & -1 & 2 \\ 0 & -5 & 3 & -2 \\ 0 & 0 & -\frac{11}{5} & -\frac{11}{5} \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -1 & 2 \\ 0 & -5 & 3 & -2 \\ 0 & 0 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

- The vector in the "border" is the solution