Lesson 6: Linear independence, matrix column space and null space

- New concepts:
  - Span of a vector set, matrix column space (range)
  - Linearly dependent set of vectors
  - Matrix null space

Two linear systems: same system matrix, different right hand side

$$\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix} \in \mathbb{R}^{3\times 3}, \boldsymbol{b} = \begin{pmatrix} 3 \\ 1 \\ 3 \end{pmatrix} \in \mathbb{R}^3, \boldsymbol{c} = \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} \in \mathbb{R}^3 \tag{1}$$

$$\begin{cases} x_1 + 2x_2 + 3x_3 &= 3 \\ \qquad x_2 + x_3 &= 1 \\ x_1 + 2x_2 + 3x_3 &= 3 \end{cases} \Leftrightarrow \boldsymbol{Ax} = \boldsymbol{b}, \begin{cases} y_1 + 2y_2 + 3y_3 &= 3 \\ \qquad y_2 + y_3 &= 1 \\ y_1 + 2y_2 + 3y_3 &= 4 \end{cases} \Leftrightarrow \boldsymbol{Ay} = \boldsymbol{c}$$

Form the bordered matrix in both cases, and reduce to triangular form (Gaussian elimination)

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 3 \\ 0 & 1 & 1 & 1 \\ 1 & 2 & 3 & 3 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 2 & 3 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \Leftrightarrow \begin{cases} x_1 + 2x_2 + 3x_3 &= 3 \\ \qquad x_2 + x_3 &= 1 \quad \text{Infinite number of solutions} \\ \qquad\qquad 0 &= 0 \end{cases}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 3 \\ 0 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 2 & 3 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right) \Leftrightarrow \begin{cases} y_1 + 2y_2 + 2y_3 &= 3 \\ \qquad y_2 &= 1 \quad \text{No solutions} \\ \qquad\qquad 0 &\neq 1 \end{cases}$$

Recall: Solving $\boldsymbol{A}\boldsymbol{x}=\boldsymbol{b}$ means finding the linear combination of columns of $\boldsymbol{A}$ such that

$$x_1\boldsymbol{a}_1+x_2\boldsymbol{a}_2+x_3\boldsymbol{a}_3=\boldsymbol{b}, \boldsymbol{x}=\begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}^T \tag{2}$$

For previous examples

$$\boldsymbol{A}=\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix} \Rightarrow \boldsymbol{a}_1=\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \boldsymbol{a}_2=\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}, \boldsymbol{a}_3=\begin{pmatrix} 3 \\ 1 \\ 3 \end{pmatrix}$$

Note that $\boldsymbol{a}_3=\boldsymbol{a}_1+\boldsymbol{a}_2$, so the linear combination (2) can be rewritten as

$$(x_1+x_3)\boldsymbol{a}_1+(x_2+x_3)\boldsymbol{a}_2=\boldsymbol{b}$$

If $\boldsymbol{b}$ is in the plane defined by the two directions $\boldsymbol{a}_1, \boldsymbol{a}_2$ then there are an infinity of choices of $\boldsymbol{x}=\begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}^T$ to obtain $\boldsymbol{b}$ by the linear combination of $\boldsymbol{a}_1, \boldsymbol{a}_2$. This is the first example.

Since $\boldsymbol{c}=\begin{pmatrix} 3 & 1 & 4 \end{pmatrix}^T$ is not in the planed defined by $\boldsymbol{a}_1, \boldsymbol{a}_2$, there is no linear combination of $\boldsymbol{a}_1, \boldsymbol{a}_2$ to obtain $\boldsymbol{c}$, hence there is no solution to the system $\boldsymbol{A}\boldsymbol{y}=\boldsymbol{c}$.

- Generalize this idea of "vectors reachable by linear combination of other vectors"

Definition. *The span of vectors $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n \in \mathcal{V}$, is the set of vectors reachable by linear combination*

$$\operatorname{span}\{\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n\} = \{\boldsymbol{b} \in \mathcal{V} \,|\, \exists x_1, \ldots, x_n \in \mathcal{S} \text{ such that } \boldsymbol{b} = x_1 \boldsymbol{a}_1 + \ldots x_n \boldsymbol{a}_n\}.$$

The notation used for set on the right hand side is read: "those vectors $\boldsymbol{b}$ in $\mathcal{V}$ with the property that there exist n scalars $x_1, \ldots, x_n$ to obtain $\boldsymbol{b}$ by linear combination of $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n$.

- A linear combination is conveniently expressed as a matrix-vector product leading to a different formulation of the same concept

Definition. *The column space (or range) of matrix $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ is the set of vectors reachable by linear combination of the matrix column vectors*

$$C(\boldsymbol{A}) = \operatorname{range}(\boldsymbol{A}) = \{\boldsymbol{b} \in \mathbb{R}^m \,|\, \exists \boldsymbol{x} \in \mathbb{R}^n \text{ such that } \boldsymbol{b} = \boldsymbol{A}\boldsymbol{x}\} \subseteq \mathbb{R}^m$$

In the example (1)

$$\boldsymbol{A} = \begin{pmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \boldsymbol{a}_3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix}$$

$$\operatorname{span}\{\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3\} = \operatorname{span}\{\boldsymbol{a}_1, \boldsymbol{a}_2\}$$

since $\boldsymbol{a}_3 = \boldsymbol{a}_1 + \boldsymbol{a}_2 \Leftrightarrow \boldsymbol{a}_1 + \boldsymbol{a}_2 - \boldsymbol{a}_3 = \boldsymbol{0}$. Introduce a concept to capture the idea that a vector can be expressed in terms of other vectors.

Definition. *The vectors $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, ..., $\boldsymbol{a}_n \in \mathcal{V}$, are linearly dependent if there exist n scalars, $x_1, \ldots, x_n \in \mathcal{S}$, at least one of which is different from zero such that*

$$x_1 \boldsymbol{a}_1 + \ldots x_n \boldsymbol{a}_n = \boldsymbol{0}$$

Note that $\{\boldsymbol{0}\}$, with $\boldsymbol{0} \in \mathcal{V}$ is a linearly dependent set of vectors since $1 \cdot \boldsymbol{0} = \boldsymbol{0}$.

The converse of linear dependence is linear independence, a member of the set cannot be expressed as a non-trivial linear combination of the other vectors

Definition. *The vectors $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n \in \mathcal{V}$, are linearly independent if the only n scalars, $x_1, \ldots, x_n \in \mathcal{S}$, that satisfy*

$$x_1 \boldsymbol{a}_1 + \ldots x_n \boldsymbol{a}_n = \mathbf{0}, \tag{3}$$

are $x_1 = 0$, $x_2 = 0, \ldots, x_n = 0$.

The choice $\boldsymbol{x} = \begin{pmatrix} x_1 & \ldots & x_n \end{pmatrix}^T = \mathbf{0}$ that always satisfies (3) is called a *trivial solution*. We can restate linear independence as (3) being satisfied *only* by the trivial solution.

Introduce a characterization of the column vectors of a matrix related to linear dependence

Definition. *The* *null space* *of a matrix* $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ *is the set*

$$N(\boldsymbol{A}) = \mathrm{null}(\boldsymbol{A}) = \{\boldsymbol{x} \in \mathbb{R}^n \,|\, \boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}\} \subseteq \mathbb{R}^n$$

- If $\mathrm{null}(\boldsymbol{A}) = \{\boldsymbol{0}\}$ then the column vectors of $\boldsymbol{A}$ are linearly independent, since the only way to satisfy (3) is by the trivial solution $\boldsymbol{x} = \boldsymbol{0}$

For example (1) we have $c\,(\boldsymbol{a}_1 + \boldsymbol{a}_2 - \boldsymbol{a}_3) = \boldsymbol{0}$ for any scalar c, hence

$$\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix} \Rightarrow \boldsymbol{a}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \boldsymbol{a}_2 = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}, \boldsymbol{a}_3 = \begin{pmatrix} 3 \\ 1 \\ 3 \end{pmatrix}$$

$$C(\boldsymbol{A}) = \mathrm{span}\{\boldsymbol{a}_1, \boldsymbol{a}_2\}, N(\boldsymbol{A}) = \mathrm{span}\left\{\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}\right\}$$