- New concepts:
  - Row space
  - Basis for a vector space
  - Dimension of a vector space
  - Sum, direct sum, and intersection of vector spaces
  - Orthogonal subspaces, orthogonal complements
  - Fundamental theorem of linear algebra (FTLA) for $\boldsymbol{A} \in \mathbb{R}^{m \times n}$:

$$C(\boldsymbol{A}) \oplus N(\boldsymbol{A}^T) = \mathbb{R}^m, \;\; C(\boldsymbol{A}) \perp N(\boldsymbol{A}^T), \;\; C(\boldsymbol{A}) \cap N(\boldsymbol{A}^T) = \{\mathbf{0}\},$$
$$C(\boldsymbol{A}^T) \oplus N(\boldsymbol{A}) = \mathbb{R}^n, \;\; C(\boldsymbol{A}^T) \perp N(\boldsymbol{A}), \;\; C(\boldsymbol{A}^T) \cap N(\boldsymbol{A}) = \{\mathbf{0}\}.$$

- For $\boldsymbol{A} \in \mathbb{R}^{m \times n}$, seen as a linear mapping $\boldsymbol{A}\colon \mathbb{R}^n \mapsto \mathbb{R}^m$, that given input vector $\boldsymbol{x} \in \mathbb{R}^n$ returns output vector $\boldsymbol{b} \in \mathbb{R}^m$, $\boldsymbol{b} = \boldsymbol{A}\boldsymbol{x}$, we have defined the vector space of possible outputs, the column space of $\boldsymbol{A}$

$$C(\boldsymbol{A}) = \{\boldsymbol{b} \in \mathbb{R}^m \,|\, \exists \boldsymbol{x} \in \mathbb{R}^n \,\text{such that}\, \boldsymbol{b} = \boldsymbol{A}\boldsymbol{x}\} \subseteq \mathbb{R}^m$$

- The transpose $\boldsymbol{A}^T \in \mathbb{R}^{n \times m}$ can also be seen as a linear mapping. Given some input vector $\boldsymbol{y} \in \mathbb{R}^m$ the mapping returns the output vector $\boldsymbol{c} \in \mathbb{R}^n$, $\boldsymbol{c} = \boldsymbol{A}^T\boldsymbol{y}$. The set of possible outputs is the column space of $\boldsymbol{A}^T$. Since columns of $\boldsymbol{A}^T$ are rows of $\boldsymbol{A}$, we can define the *row space* of $\boldsymbol{A}$ as

$$R(\boldsymbol{A}) = C(\boldsymbol{A}^T) = \{\boldsymbol{c} \in \mathbb{R}^n \,|\, \exists \boldsymbol{y} \in \mathbb{R}^m \,\text{such that}\, \boldsymbol{c} = \boldsymbol{A}^T\boldsymbol{y}\} \subseteq \mathbb{R}^n$$

Basis for a vector space

Definition. *A set of vectors* $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n \in \mathcal{V}$ *is a basis for vector space* $\mathcal{V}$ *if:*

1. $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n$ *are linearly independent;*
2. $\operatorname{span}\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n\} = \mathcal{V}$.

Definition. *The number of vectors* $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n \in \mathcal{V}$ *within a basis is the dimension of the vector space* $\mathcal{V}$.

Definition. *Given two vector subspaces* $(\mathcal{U},\mathcal{S},+)$, $(\mathcal{V},\mathcal{S},+)$ *of the space* $(\mathcal{W},\mathcal{S},+)$, *the sum is the set* $\mathcal{U}+\mathcal{V}=\{\boldsymbol{u}+\boldsymbol{v}|\ \boldsymbol{u}\in\mathcal{U},\boldsymbol{v}\in\mathcal{V}\}$.

Definition. *Given two vector subspaces* $(\mathcal{U},\mathcal{S},+)$, $(\mathcal{V},\mathcal{S},+)$ *of the space* $(\mathcal{W},\mathcal{S},+)$, *the direct sum is the set* $\mathcal{U}\oplus\mathcal{V}=\{\boldsymbol{u}+\boldsymbol{v}|\ \exists!\boldsymbol{u}\in\mathcal{U},\exists!\boldsymbol{v}\in\mathcal{V}\}$. *(unique decomposition)*

Definition. *Given two vector subspaces* $(\mathcal{U},\mathcal{S},+)$, $(\mathcal{V},\mathcal{S},+)$ *of the space* $(\mathcal{W},\mathcal{S},+)$, *the intersection is the set*

$$\mathcal{U}\cap\mathcal{V}=\{\boldsymbol{x}|\boldsymbol{x}\in\mathcal{U},\boldsymbol{x}\in\mathcal{V}\}.$$

Definition. *Two vector subspaces* $(\mathcal{U},\mathcal{S},+)$, $(\mathcal{V},\mathcal{S},+)$ *of the space* $(\mathcal{W},\mathcal{S},+)$ *are orthogonal subspaces, denoted* $\mathcal{U}\perp\mathcal{V}$ *if* $\boldsymbol{u}^T\boldsymbol{v}=0$ *for any* $\boldsymbol{u}\in\mathcal{U},\boldsymbol{v}\in\mathcal{V}$.

Definition. *Two vector subspaces* $(\mathcal{U},\mathcal{S},+)$, $(\mathcal{V},\mathcal{S},+)$ *of the space* $(\mathcal{W},\mathcal{S},+)$ *are orthogonal complements, denoted* $\mathcal{U}=\mathcal{V}^\perp$, $\mathcal{V}=\mathcal{U}^\perp$ *if they are orthogonal subspaces and* $\mathcal{U}\cap\mathcal{V}=\{\mathbf{0}\}$, *i.e., the null vector is the only common element of both subspaces.*

The four fundamental subspaces for a linear mapping

- A matrix $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ is a linear mapping from $\mathbb{R}^n$ to $\mathbb{R}^m$, $\boldsymbol{A}: \mathbb{R}^n \to \mathbb{R}^m$
- The transpose $\boldsymbol{A}^T \in \mathbb{R}^{n \times m}$ is a linear mapping from $\mathbb{R}^m$ to $\mathbb{R}^n$, $\boldsymbol{A}^T: \mathbb{R}^m \to \mathbb{R}^n$
- To each matrix $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ associate four fundamental subspaces:
  1. *Column space*, $C(\boldsymbol{A}) = \{\boldsymbol{b} \in \mathbb{R}^m \,|\, \exists \boldsymbol{x} \in \mathbb{R}^n \text{ such that } \boldsymbol{b} = \boldsymbol{A}\boldsymbol{x}\} \subseteq \mathbb{R}^m$, the part of $\mathbb{R}^m$ *reachable* by linear combination of columns of $\boldsymbol{A}$
  2. *Left null space*, $N(\boldsymbol{A}^T) = \{\boldsymbol{y} \in \mathbb{R}^m \,|\, \boldsymbol{A}^T\boldsymbol{y} = 0\} \subseteq \mathbb{R}^m$, the part of $\mathbb{R}^m$ *not reachable* by linear combination of columns of $\boldsymbol{A}$
  3. *Row space*, $R(\boldsymbol{A}) = C(\boldsymbol{A}^T) = \{\boldsymbol{c} \in \mathbb{R}^n \,|\, \exists \boldsymbol{y} \in \mathbb{R}^m \text{ such that } \boldsymbol{c} = \boldsymbol{A}^T\boldsymbol{y}\} \subseteq \mathbb{R}^n$, the part of $\mathbb{R}^n$ *reachable* by linear combination of rows of $\boldsymbol{A}$
  4. *Null space*, $N(\boldsymbol{A}) = \{\boldsymbol{x} \in \mathbb{R}^n \,|\, \boldsymbol{A}\boldsymbol{x} = 0\} \subseteq \mathbb{R}^n$, the part of $\mathbb{R}^n$ *not reachable* by linear combination of rows of $\boldsymbol{A}$

Theorem. *Given the linear mapping associated with matrix* $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ *we have:*

1. $C(\boldsymbol{A}) \oplus N(\boldsymbol{A}^T) = \mathbb{R}^m$, *the direct sum of the column space and left null space is the codomain of the mapping*
2. $C(\boldsymbol{A}^T) \oplus N(\boldsymbol{A}) = \mathbb{R}^n$, *the direct sum of the row space and null space is the domain of the mapping*
3. $C(\boldsymbol{A}) \perp N(\boldsymbol{A}^T)$ *and* $C(\boldsymbol{A}) \cap N(\boldsymbol{A}^T) = \{\mathbf{0}\}$, *the column space is orthogonal to the left null space, and they are orthogonal complements of one another,*

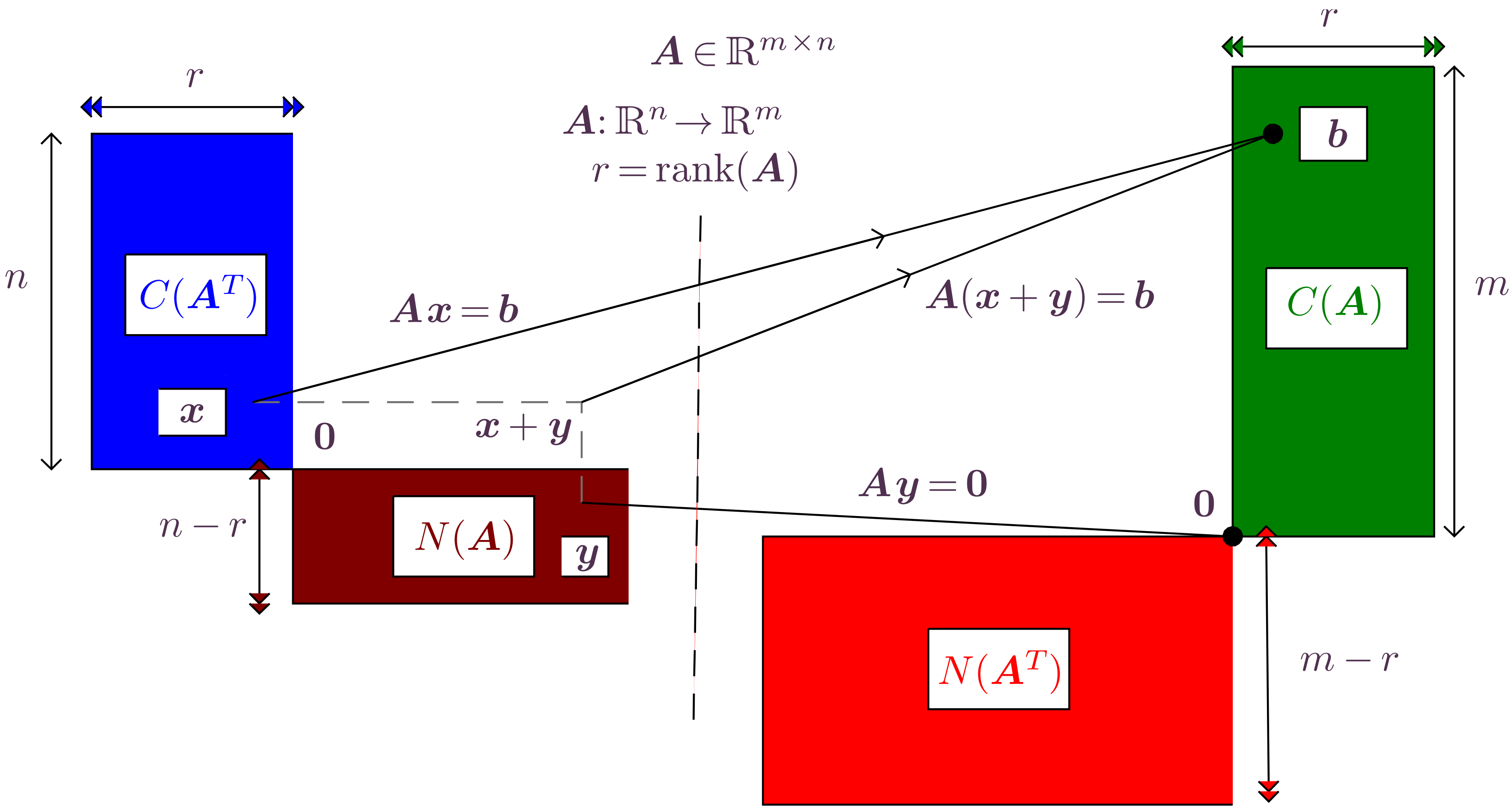
$$C(\boldsymbol{A}) = N(\boldsymbol{A}^T)^{\perp}, \;\; N(\boldsymbol{A}^T) = C(\boldsymbol{A})^{\perp} \; .$$

4. $C(\boldsymbol{A}^T) \perp N(\boldsymbol{A})$ *and* $C(\boldsymbol{A}^T) \cap N(\boldsymbol{A}) = \{\mathbf{0}\}$, *the row space is orthogonal to the null space, and they are orthogonal complements of one another,*

$$C(\boldsymbol{A}^T) = N(\boldsymbol{A})^{\perp}, \;\; N(\boldsymbol{A}) = C(\boldsymbol{A}^T)^{\perp} \; .$$

Graphical representation of FTLA

Gil Strang introduced a very useful graphical represenation in "The Fundamental Theorem of Linear Algebra." *Amer. Math. Monthly* **100**, 848-855, 1993.



$\mathbb{R}^n = C(A^T) \oplus N(A)$ usually: $m \geqslant n$ $\mathbb{R}^m = N(A^T) \oplus C(A)$

$C(A^T) \perp N(A)$ $N(A^T) \perp C(A)$