Lesson 10: Rank-nullity theorem, General solution of $\boldsymbol{A}\boldsymbol{x}=\boldsymbol{b}$ ($\boldsymbol{A}\in\mathbb{R}^{m\times m}$)

- New concepts:
  - rank-nullity theorem
  - Inverse matrix
  - Gauss-Jordan algorithm to find inverse

Definition. *The rank of a matrix* $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ *is the dimension of the column space* $r = \dim C(\boldsymbol{A})$.

Definition. *The nullity of a matrix* $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ *is the dimension of the null space* $z = \dim N(\boldsymbol{A})$.

Proposition. *The dimension of the column space is equal to the dimension of the row space.*

Corollary. *The system* $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$, $\boldsymbol{A} \in \mathbb{R}^{m \times n}$, $\boldsymbol{x} \in \mathbb{R}^{n}$, $\boldsymbol{b} \in \mathbb{R}^{m}$ *has a solution if* $\boldsymbol{b} \in \mathbb{R}^{m}$. *The solution is unique if* $N(\boldsymbol{A}) = \{\boldsymbol{0}\}$ *(the nullity of* $\boldsymbol{A}$ *is zero)*

Definition. *A square matrix has the same number of columns as rows,* $\boldsymbol{A} \in \mathbb{R}^{m \times m}$.

Definition. *A linear system with a null right hand side,* $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}$ *is said to be homogeneous.*

Definition. *The square matrix* $\boldsymbol{A} \in \mathbb{R}^{m \times m}$ *is nonsingular if the only solution to the homogeneous linear system* $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}$ *is* $\boldsymbol{x} = \boldsymbol{0} \in \mathbb{R}^m$.

Proposition. *The columns of a nonsingular matrix are linearly independent. A square matrix with linearly independent columns is nonsingular*

Proof. The column form of the matrix is $\boldsymbol{A} = \left(\begin{array}{cccc} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \ldots & \boldsymbol{a}_m \end{array} \right)$, with $\boldsymbol{a}_j \in \mathbb{R}^m$ for $j = 1, \ldots, m$. The matrix vector product $\boldsymbol{A}\boldsymbol{x}$ expresses the linear combination of column vectors

$$\boldsymbol{A}\boldsymbol{x} = x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 + \ldots + x_m\boldsymbol{a}_m.$$

If $\boldsymbol{A} \in \mathbb{R}^{m \times m}$ is nonsingular then the only solution of $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}$ is $\boldsymbol{x} = 0$ hence $\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m\}$ are linearly independent. Conversely, if $\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m\}$ are linearly independent then $x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 + \ldots + x_m\boldsymbol{a}_m = \boldsymbol{0}$ implies $\boldsymbol{x} = 0$, or $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}$, hence $\boldsymbol{A}$ nonsingular.

Nonsingular matrices row reduce to the identity matrix

- For $\boldsymbol{I} \in \mathbb{R}^{m \times m}$ the identity matrix

$$C(\boldsymbol{I}) = \mathbb{R}^m \quad N(\boldsymbol{I}^T) = \{\mathbf{0}\} \quad C(\boldsymbol{I}^T) = \mathbb{R}^m \quad N(\boldsymbol{I}) = \{\mathbf{0}\} \quad \operatorname{rank}(\boldsymbol{I}) = m$$

- For $\boldsymbol{A} \in \mathbb{R}^{m \times m}$ nonsingular

$$C(\boldsymbol{A}) = \mathbb{R}^m \quad N(\boldsymbol{A}^T) = \{\mathbf{0}\} \quad C(\boldsymbol{A}^T) = \mathbb{R}^m \quad N(\boldsymbol{A}) = \{\mathbf{0}\} \quad \operatorname{rank}(\boldsymbol{A}) = m$$

Proposition. *Let $\boldsymbol{B} \in \mathbb{R}^{m \times m}$ be the row reduced form of $\boldsymbol{A} \in \mathbb{R}^{m \times m}$. The matrix $\boldsymbol{A}$ is nonsingular if and only if (iff) $\boldsymbol{B}$ is the identity matrix.*

Proof. $(\Rightarrow)$ *$\boldsymbol{A}$ nonsingular has rank m, hence m linearly independent rows and the row reduction procedure produces $\boldsymbol{B} = \boldsymbol{I}$.*

$(\Leftarrow)$ If $\boldsymbol{B} = \boldsymbol{I}$ the row reduction of the augmented system $(\ \boldsymbol{A}\ \mathbf{0}) \sim (\ \boldsymbol{I}\ \ \mathbf{0}\ )$ with unique solution $\boldsymbol{x} = \mathbf{0}$, hence $\boldsymbol{A}$ nonsingular

Proposition. $\boldsymbol{A} \in \mathbb{R}^{m \times m}$ *nonsingular is equivalent to existence of a unique solution to* $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ *for any* $\boldsymbol{b} \in \mathbb{R}^m$.

Proof. $(\Rightarrow)$ *Row reduction of the augmented system* $(\ \boldsymbol{A}\ \boldsymbol{b}) \sim (\ \boldsymbol{I}\ \boldsymbol{c})$ *with unique solution.*

$(\Leftarrow)$ *Choose* $\boldsymbol{b} = \boldsymbol{0}$ *to obtain unique solution* $\boldsymbol{x} = \boldsymbol{0}$ *hence* $\boldsymbol{A}$ *is nonsingular.*

- Interpret the above as follows:
  - The same vector in $\mathbb{R}^m$ is expressed as $\boldsymbol{A}\boldsymbol{x}$, a linear combination of columns of $\boldsymbol{A} \in \mathbb{R}^{m \times m}$, and as a linear combination $\boldsymbol{I}\boldsymbol{b}$, of the columns of $\boldsymbol{I} \in \mathbb{R}^{m \times m}$
  - For every $\boldsymbol{x}$ we obtain a unique $\boldsymbol{b} = \boldsymbol{A}\boldsymbol{x}$
  - When $\boldsymbol{A}$ is nonsingular we obtain a unique $\boldsymbol{x}$ for every $\boldsymbol{b}$

Definition. *Given a nonsingular matrix $\boldsymbol{A} \in \mathbb{R}^{m \times m}$, the inverse of $\boldsymbol{A}$ is an $m \times m$ matrix denoted as $\boldsymbol{A}^{-1}$ that satisfies the properties*

$$\boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{I},$$

with $\boldsymbol{I}$ the $m \times m$ identity matrix.

When $\boldsymbol{A}$ is nonsingular, the solution to the linear system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$, can be expressed using the inverse as

$$\boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{b}.$$

Computation of the inverse: Gauss-Jordan

- Consider $\boldsymbol{A} \in \mathbb{R}^{m \times m}$ nonsingular. Denote the inverse of $\boldsymbol{A}$ as $\boldsymbol{X} = \boldsymbol{A}^{-1}$, $\boldsymbol{X} \in \mathbb{R}^{m \times m}$.
- The column vector form of $\boldsymbol{X}$ is $\boldsymbol{X} = \begin{pmatrix} \boldsymbol{x}_1 & \dots & \boldsymbol{x}_m \end{pmatrix}$, $\boldsymbol{x}_i \in \mathbb{R}^m$ for $i = 1, 2, \dots, m$
- By definition of the inverse $\boldsymbol{B}$ satisfies

$$\boldsymbol{A}\boldsymbol{X} = \boldsymbol{A}\begin{pmatrix} \boldsymbol{x}_1 & \dots & \boldsymbol{x}_m \end{pmatrix} = \begin{pmatrix} \boldsymbol{A}\boldsymbol{x}_1 & \dots & \boldsymbol{A}\boldsymbol{x}_m \end{pmatrix} = \boldsymbol{I} = \begin{pmatrix} \boldsymbol{e}_1 & \dots & \boldsymbol{e}_m \end{pmatrix}$$

- Finding the inverse is therefore equivalent to solving the m linear systems $\boldsymbol{A}\boldsymbol{x}_i = \boldsymbol{e}_i$
- This can be carried out by applying the row-reduction technique to the augmented matrix

$$\begin{pmatrix} \boldsymbol{A} & | & \boldsymbol{I} \end{pmatrix} \sim \begin{pmatrix} \boldsymbol{I} & | & \boldsymbol{X} \end{pmatrix}$$

- By carrying out steps to obtain the identity matrix in the left half, the matrix resulting in the right half is the inverse matrix, $\boldsymbol{X} = \boldsymbol{A}^{-1}$

Gauss-Jordan example (using Octave)

Apply the Gauss-Jordan algorithm to find the inverse of

$$\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 1 \\ -1 & 0 & 2 \\ 2 & -1 & -4 \end{pmatrix}$$

```
octave> A=[1 2 1; -1 0 2; 2 -1 -4]; AX=[A eye(3)]; format rat; disp(AX);
```

```
   1    2    1    1    0    0
  -1    0    2    0    1    0
   2   -1   -4    0    0    1
```

```
octave> AX(2,:)=AX(2,:)+AX(1,:); AX(3,:)=AX(3,:)-2*AX(1,:); disp(AX);
```

```
   1    2    1    1    0    0
   0    2    3    1    1    0
   0   -5   -6   -2    0    1
```

```
octave> AX(2,:)=(1/2)*AX(2,:); disp(AX);
```

```
   1    2    1    1    0    0
   0    1  3/2  1/2  1/2    0
   0   -5   -6   -2    0    1
```

```
octave>
```

Gauss-Jordan example (continued)

```
octave> AX(3,:)=AX(3,:)+5*AX(2,:); disp(AX);
```

1	2	1	1	0	0
0	1	3/2	1/2	1/2	0
0	0	3/2	1/2	5/2	1

```
octave> AX(3,:)=(2/3)*AX(3,:); disp(AX);
```

1	2	1	1	0	0
0	1	3/2	1/2	1/2	0
0	0	1	1/3	5/3	2/3

```
octave> AX(2,:)=AX(2,:)-(3/2)*AX(3,:); AX(1,:)=AX(1,:)-AX(3,:); disp(AX);
```

1	2	0	2/3	-5/3	-2/3
0	1	0	0	-2	-1
0	0	1	1/3	5/3	2/3

```
octave> AX(1,:)=AX(1,:)-2*AX(2,:); X=AX(:,4:6); disp(AX);
```

1	0	0	2/3	7/3	4/3
0	1	0	0	-2	-1
0	0	1	1/3	5/3	2/3

```
octave> disp([A*X X*A]);
```

1	0	0	1	0	0
0	1	0	0	1	0
0	0	1	0	0	1

```
octave>
```