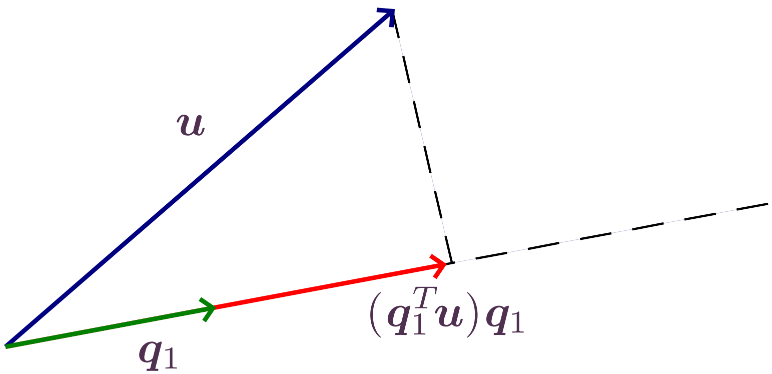
Lesson 12: Projection

- New concepts:
  - Projector matrix, complementary projector
  - Orthogonal projection

- Consider a vector $\boldsymbol{u} \in \mathbb{R}^m$, and a unit-norm vector $\boldsymbol{q}_1 \in \mathbb{R}^m$

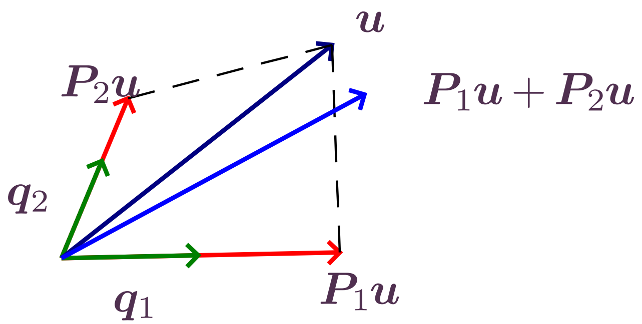


Definition. *The* orthogonal projection *of $\boldsymbol{u} \in \mathbb{R}^m$ along direction $\boldsymbol{q}_1 \in \mathbb{R}^m$, $\|\boldsymbol{q}_1\| = 1$ is the vector $(\boldsymbol{q}_1^T\boldsymbol{u})\boldsymbol{q}_1$.*

- Scalar-vector multiplication commutativity: $(\boldsymbol{q}_1^T\boldsymbol{u})\boldsymbol{q}_1 = \boldsymbol{q}_1(\boldsymbol{q}_1^T\boldsymbol{u})$
- Matrix multiplication associativity: $\boldsymbol{q}_1(\boldsymbol{q}_1^T\boldsymbol{u}) = (\boldsymbol{q}_1\boldsymbol{q}_1^T)\boldsymbol{u} = \boldsymbol{P}_1\boldsymbol{u}$, with $\boldsymbol{P}_1 \in \mathbb{R}^{m\times m}$

Definition. *The matrix $\boldsymbol{P}_1 = \boldsymbol{q}_1\boldsymbol{q}_1^T \in \mathbb{R}^{m\times m}$ is the* orthogonal projector *along direction $\boldsymbol{q}_1 \in \mathbb{R}^m$, $\|\boldsymbol{q}_1\| = 1$.*

- Consider n orthonormal vectors grouped into a matrix $\boldsymbol{Q} = \begin{pmatrix} \boldsymbol{q}_1 & \dots & \boldsymbol{q}_n \end{pmatrix} \in \mathbb{R}^{m \times n}$



- The orthogonal projection of $\boldsymbol{u}$ onto the subspace spanned by $\boldsymbol{q}_1, \dots, \boldsymbol{q}_n$ is

$$\boldsymbol{P}\boldsymbol{u} = \boldsymbol{P}_1\boldsymbol{u} + \dots + \boldsymbol{P}_n\boldsymbol{u} = (\boldsymbol{q}_1\boldsymbol{q}_1^T)\boldsymbol{u} + \dots + (\boldsymbol{q}_n\boldsymbol{q}_n^T)\boldsymbol{u} \Rightarrow$$

$$\boldsymbol{P} = \boldsymbol{q}_1\boldsymbol{q}_1^T + \dots + \boldsymbol{q}_n\boldsymbol{q}_n^T = \begin{pmatrix} \boldsymbol{q}_1 & \dots & \boldsymbol{q}_n \end{pmatrix} \begin{pmatrix} \boldsymbol{q}_1^T \\ \vdots \\ \boldsymbol{q}_n^T \end{pmatrix} = \boldsymbol{Q}\boldsymbol{Q}^T$$

Definition. *The orthogonal projector onto $C(\boldsymbol{Q})$, $\boldsymbol{Q} \in \mathbb{R}^{m \times n}$ with orthonormal column vectors is $\boldsymbol{P} = \boldsymbol{Q}\boldsymbol{Q}^T$*

Complementary orthogonal projector

- Given $\boldsymbol{u} \in \mathbb{R}^m$ and $\boldsymbol{Q} = \begin{pmatrix} \boldsymbol{q}_1 & \dots & \boldsymbol{q}_n \end{pmatrix} \in \mathbb{R}^{m \times n}$ with orthonormal columns

Definition. *The complementary orthogonal projector to $\boldsymbol{P} = \boldsymbol{Q}\boldsymbol{Q}^T$ is $\boldsymbol{I} - \boldsymbol{P}$, where $\boldsymbol{Q} \in \mathbb{R}^{m \times n}$ is a matrix with orthonormal columns.*

- The complementary orthogonal projector projects a vector onto the left null space, $N(\boldsymbol{Q}^T)$

- Consider the linear system $\boldsymbol{A}\boldsymbol{x}=\boldsymbol{b}$ with $\boldsymbol{A}\in\mathbb{R}^{m\times n}$, $\boldsymbol{x}\in\mathbb{R}^n$, $\boldsymbol{b}\in\mathbb{R}^m$. Orthogonal projectors and knowledge of the four fundamental matrix subspaces allows us to succintly express whether there exist no solutions, a single solution of an infinite number of solutions:
  - Consider the factorization $\boldsymbol{Q}\boldsymbol{R}=\boldsymbol{A}$, the orthogonal projector $\boldsymbol{P}=\boldsymbol{Q}\boldsymbol{Q}^T$, and the complementary orthogonal projector $\boldsymbol{I}-\boldsymbol{P}$
  - If $\|(\boldsymbol{I}-\boldsymbol{P})\boldsymbol{b}\|\neq 0$, then $\boldsymbol{b}$ has a component outside the column space of $\boldsymbol{A}$, and $\boldsymbol{A}\boldsymbol{x}=\boldsymbol{b}$ has no solution
  - If $\|(\boldsymbol{I}-\boldsymbol{P})\boldsymbol{b}\|=0$, then $\boldsymbol{b}\in C(\boldsymbol{Q})=C(\boldsymbol{A})$ and the system has at least one solution
  - If $N(\boldsymbol{A})=\{\boldsymbol{0}\}$ (null space only contains the zero vector, i.e., null space of dimension 0) the system has a unique solution
  - If $\dim N(\boldsymbol{A})=n-r>0$, then a vector $\boldsymbol{y}\in N(\boldsymbol{A})$ in the null space is written as

$$\boldsymbol{y}=c_1\boldsymbol{z}_1+\ldots+c_{n-r}\boldsymbol{z}_{n-r}$$

    and if $\boldsymbol{x}$ is a solution of $\boldsymbol{A}\boldsymbol{x}=\boldsymbol{b}$, so is $\boldsymbol{x}+\boldsymbol{y}$, since

$$\boldsymbol{A}(\boldsymbol{x}+\boldsymbol{y})=\boldsymbol{A}\boldsymbol{x}+c_1\boldsymbol{A}\boldsymbol{z}_1+\ldots+c_{n-r}\boldsymbol{A}\boldsymbol{z}_{n-r}=\boldsymbol{b}+\boldsymbol{0}+\ldots+\boldsymbol{0}=\boldsymbol{b}$$

    The linear system has an $(n-r)$-parameter family of solutions