- New concepts
  - projection in an inner product space
  - linear spaces of functions
  - Gram-Schmidt algorithm in an inner product space
  - Fourier analysis

Recall the definition introduced in Lesson 4

Definition. *Consider vectors* $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \mathcal{V}$ *and scalar* $a \in S$. *The function*

$$\langle\,,\rangle : \mathcal{V} \times \mathcal{V} \to S$$

is an inner product if:

1. $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \overline{\langle \boldsymbol{v}, \boldsymbol{u} \rangle}$ *(Conjugate symmetry)*
2. $\langle a\boldsymbol{u}, \boldsymbol{v} \rangle = a\langle \boldsymbol{u}, \boldsymbol{v} \rangle, \langle \boldsymbol{u} + \boldsymbol{v}, \boldsymbol{w} \rangle = \langle \boldsymbol{u}, \boldsymbol{w} \rangle + \langle \boldsymbol{v}, \boldsymbol{w} \rangle$ *(Linearity in first argument)*
3. $\langle \boldsymbol{u}, \boldsymbol{u} \rangle \geqslant 0, \langle \boldsymbol{u}, \boldsymbol{u} \rangle = 0 \Rightarrow \boldsymbol{u} = \boldsymbol{0}$ *(Positive definiteness)*

This definition is constructed to remain valid for complex scalars $S = \mathbb{C}$. Recall the conjugate of a complex number $z = x + iy$ is $\bar{z} = x - iy$. The dot product $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \boldsymbol{u}^T \boldsymbol{v}$, between vectors $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^m$ is the most familiar example of an inner product. The term *scalar product* is a synonym for *inner product*.

- Recall that a function consists of a two sets X, Y (domain, codomain), and a procedure to associate just one element in Y to an element in X, $f: X \to Y$. The value of the function for $x \in X$ is often denoted as $y = f(x) \in Y$. The term "associate" is imprecise, hence the following technical definitions

Definition. *The Cartesian product of two sets X, Y is the set of ordered pairs $X \times Y = \{(x, y) | x \in X, y \in Y\}$.*

Definition. *A function defined on non-empty sets X, Y, $f: X \to Y$, is a subset of $X \times Y$ such that $\forall x \in X$, $\exists! y = f(x) \in Y$.*

- A vector with m real components, $\boldsymbol{u} \in \mathbb{R}^m$ is also a function $u: \{1, \ldots, m\} \to \mathbb{R}$, defined by the set of ordered pairs $\{(1, u_1), (2, u_2), \ldots, (m, u_m)\}$
- The concept of a linear space and operations in a linear space that have been so far been considered for vectors, readily generalize to linear spaces of functions, with myriad applications throughout quantitative science.

Restate the definition from Lesson 7, replacing the notation $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ indicative of $\mathcal{V} = \mathbb{R}^m$, with $f, g, h \in \mathcal{V}$, for some general $\mathcal{V}$.

Definition. $(\mathcal{V}, \mathcal{S}, +)$ *is a linear space if for any* $f, g, h \in \mathcal{V}$, *and any* $\alpha, \beta \in \mathcal{S}$, *with* $\mathcal{S}$ *a scalar field, the following properties hold:*

Closed. $f + g \in \mathcal{V}$

Associativity. $f + (g + h) = (f + g) + h$

Null element. $\exists 0 \in \mathcal{V}$ *such that* $f + 0 = f$

Inverse element. $\exists(-f)$ *such that* $f + (-f) = \mathbf{0}$

Commutativity. $f + g = g + f$

Distributivity over scalar addition. $(\alpha + \beta) f = \alpha f + \beta g$

Distributivity over vector addition. $\alpha(f + g) = \alpha f + \alpha g$

Scalar identity. $1 \in \mathcal{S} \Rightarrow 1 f = f$

The close analogy allows us to work with functions, much the same way we work with vectors.

- $C[a,b] = \{f \mid f\colon [a,b] \to \mathbb{R},\, f \text{ continuous}\}$
- $C^\infty(\mathbb{R}) = \{f \mid f\colon \mathbb{R} \to \mathbb{R},\, f \text{ infinitely differentiable}\}$
- space of piecewise continuous functions
- $P_T = \{f \mid f\colon \mathbb{R} \to \mathbb{R},\, f(t+T) = f(t),\, f \text{ piecewise continuous}\}$, space of periodic piecewise continuous functions, with period T

We would like to carry out operations defined for vectors (e.g., projection) in the linear function spaces. This can be accomplished if an inner product space is defined for the linear space of functions.

- $\langle f, g\rangle = \int_a^b f(t)\, g(t)\, \mathrm{d}t$ is an inner product for $C[a,b]$
- $\langle f, g\rangle = \frac{2}{T}\int_0^T f(t)\, g(t)\, \mathrm{d}t$ is an inner product for P_T

The norm of a function, or orthogonality between two functions are subsequently defined through the inner product

$$\| f \| = \sqrt{\langle f, f\rangle},\quad f \perp g \Leftrightarrow \langle f, g\rangle = 0$$