- New concepts:
  - Permutation matrix
  - Gaussian multiplier matrix
  - $\boldsymbol{LU}$ factorization

- Denote a permutation by

$$\sigma = \begin{pmatrix} 1 & 2 & \dots & m \\ i_1 & i_2 & \cdots & i_m \end{pmatrix}$$

with $i_1, \dots, i_m \in \{1, \dots, m\}$, $i_j \neq i_k$ for $j \neq k$

- The sign of a permutation, $\nu(\sigma)$ is the number of pair swaps needed to obtain the permutation starting from the identity permutation

$$\begin{pmatrix} 1 & 2 & \dots & m \\ 1 & 2 & \cdots & m \end{pmatrix}$$

- A permutation can be specified by a permutation matrix $\boldsymbol{P}$

- Recall the basic operation in row echelon reduction: constructing a linear combination of rows to form zeros beneath the main diagonal, e.g.

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ a_{31} & a_{32} & \dots & a_{3m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mm} \end{pmatrix} \sim \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ 0 & a_{22} - \frac{a_{21}}{a_{11}} a_{12} & \dots & a_{2m} - \frac{a_{21}}{a_{11}} a_{1m} \\ 0 & a_{32} - \frac{a_{31}}{a_{11}} a_{12} & \dots & a_{3m} - \frac{a_{31}}{a_{11}} a_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a_{m2} - \frac{a_{m1}}{a_{11}} a_{12} & \dots & a_{mm} - \frac{a_{m1}}{a_{11}} a_{1m} \end{pmatrix}$$

- This can be stated as a matrix multiplication operation, with $l_{i1} = a_{i1} / a_{11}$

$$\begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ -l_{21} & 1 & 0 & \dots & 0 \\ -l_{31} & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -l_{m1} & 0 & 0 & \dots & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ a_{31} & a_{32} & \dots & a_{3m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mm} \end{pmatrix} =$$

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ 0 & a_{22} - l_{21} a_{12} & \dots & a_{2m} - l_{21} a_{1m} \\ 0 & a_{32} - l_{31} a_{12} & \dots & a_{3m} - l_{31} a_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a_{m2} - l_{m1} a_{12} & \dots & a_{mm} - l_{m1} a_{1m} \end{pmatrix}$$

Definition. *The matrix*

$$\boldsymbol{L}_k = \begin{pmatrix} 1 & \dots & 0 & \dots & 1 \\ 0 & \ddots & 0 & \dots & 0 \\ 0 & \dots & 1 & \dots & 0 \\ 0 & \dots & -l_{k+1,k} & \dots & 0 \\ 0 & \dots & -l_{k+2,k} & \dots & 0 \\ \vdots & \dots & \vdots & \ddots & \vdots \\ 0 & \dots & -l_{m,k} & \dots & 1 \end{pmatrix}$$

with $l_{i,k} = a_{i,k}^{(k)} \,/\, a_{k,k}^{(k)}$*, and* $\boldsymbol{A}^{(k)} = \left(a_{i,j}^{(k)}\right)$ *the matrix obtained after step* k *of row echelon reduction (or, equivalently, Gaussian elimination) is called a Gaussian multiplier matrix.*

- For $\boldsymbol{A} \in \mathbb{R}^{m \times m}$ nonsingular, the successive steps in row echelon reduction (or Gaussian elimination) correspond to successive multiplications on the left by Gaussian multiplier matrices

$$\boldsymbol{L}_{m-1}\boldsymbol{L}_{m-2}\ldots\boldsymbol{L}_2\boldsymbol{L}_1\boldsymbol{A} = \boldsymbol{U}$$

- The inverse of a Gaussian multiplier is

$$\boldsymbol{L}_k^{-1} = \begin{pmatrix} 1 & \ldots & 0 & \ldots & 1 \\ 0 & \ddots & 0 & \ldots & 0 \\ 0 & \ldots & 1 & \ldots & 0 \\ 0 & \ldots & l_{k+1,k} & \ldots & 0 \\ 0 & \ldots & l_{k+2,k} & \ldots & 0 \\ \vdots & \ldots & \vdots & \ddots & \vdots \\ 0 & \ldots & l_{m,k} & \ldots & 1 \end{pmatrix} = \boldsymbol{I} - (\boldsymbol{L}_k - \boldsymbol{I})$$

- From $(\boldsymbol{L}_{m-1}\boldsymbol{L}_{m-2}...\boldsymbol{L}_2\boldsymbol{L}_1)\boldsymbol{A}=\boldsymbol{U}$ obtain

$$\boldsymbol{A}=(\boldsymbol{L}_{m-1}\boldsymbol{L}_{m-2}...\boldsymbol{L}_2\boldsymbol{L}_1)^{-1}\boldsymbol{U}=\boldsymbol{L}_1^{-1}\boldsymbol{L}_2^{-1}\cdot...\cdot\boldsymbol{L}_{m-1}^{-1}\boldsymbol{U}=\boldsymbol{L}\boldsymbol{U}$$

- Due to the simple form of $\boldsymbol{L}_k^{-1}$ the matrix $\boldsymbol{L}$ is easily obtained as

$$\boldsymbol{L}=\begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ l_{2,1} & 1 & 0 & \dots & 0 & 0 \\ l_{3,1} & l_{3,2} & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ l_{m-1,1} & l_{m-1,2} & l_{m-1,3} & \dots & 1 & 0 \\ l_{m,1} & l_{m,2} & l_{m,3} & \dots & l_{m,m-1} & 1 \end{pmatrix}$$

LU factorization formulation of linear system solution

- Using the concept of an LU factorization, finding the solution to a linear system $Ax = b$ can be formulated as the following steps:
  1. Find the LU factorization, $LU = A$
  2. Replace factorization into system and regroup $Ax = b \Leftrightarrow (LU)x = b \Leftrightarrow Ly = b$. Solve the lower triangular $Ly = b$ system to find y
  3. Solve the upper triangular system $Ux = y$ to find x
- The above formulation of the steps within Gaussian elimination is very useful in computer calculations