- $\boldsymbol{A} \in \mathbb{R}^{m \times m}$ a square matrix, $\det(\boldsymbol{A}) \in \mathbb{R}$ is the oriented volume enclosed by the column vectors of $\boldsymbol{A}$ (a parallelipiped)
- Geometric interpretation of determinants
- Determinant calculation rules
- Algebraic definition of a determinant

Definition. *The determinant of a square matrix* $\boldsymbol{A} = \begin{pmatrix} \boldsymbol{a}_1 & \dots & \boldsymbol{a}_m \end{pmatrix} \in \mathbb{R}^{m\times m}$ *is a real number*

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mm} \end{vmatrix} \in \mathbb{R}$$

giving the (oriented) volume of the parallelepiped spanned by matrix column vectors.

- $m = 2$

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \det(\boldsymbol{A}) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

- $m = 3$

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \det(\boldsymbol{A}) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

- Computation of a determinant with $m = 2$

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}\,a_{21}$$

- Computation of a determinant with $m = 3$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{21}a_{32}a_{13} + a_{31}a_{12}a_{23}$$
$$-a_{13}a_{22}a_{31} - a_{23}a_{32}a_{11} - a_{33}a_{12}a_{21}$$

- Where do these determinant computation rules come from? Two viewpoints
  - *Geometric viewpoint*: determinants express parallelepiped volumes
  - *Algebraic viewpoint*: determinants are computed from all possible products that can be formed from choosing a factor from each row and each column

Determinant in 2D gives area of parallelogram

- $m = 2$

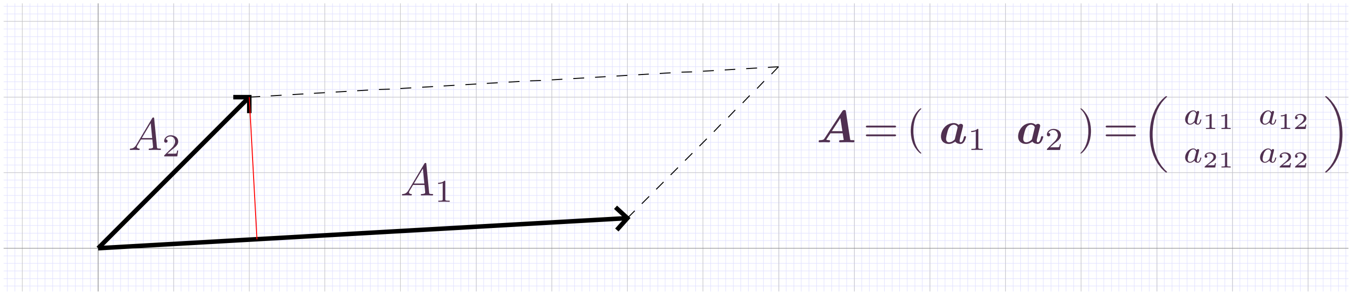


Figure 1.

- In two dimensions a "parallelepiped" becomes a parallelogram with area given as

$$(\text{Area}) = (\text{Length of Base}) \times (\text{Length of Height})$$

- Take $\boldsymbol{a}_1$ as the base, with length $b = \|\boldsymbol{a}_1\|$. Vector $\boldsymbol{a}_1$ is at angle φ_1 to x_1-axis, $\boldsymbol{a}_2$ is at angle φ_2 to x_2-axis, and the angle between $\boldsymbol{a}_1$, $\boldsymbol{a}_2$ is $\theta = \varphi_2 - \varphi_1$. The height has length

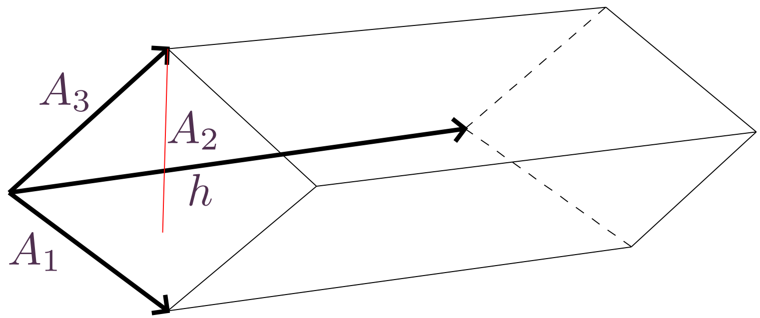
$$h = \|\boldsymbol{a}_2\| \sin\theta = \|\boldsymbol{a}_2\| \sin(\varphi_2 - \varphi_1) = \|\boldsymbol{a}_2\| (\sin\varphi_2 \cos\varphi_1 - \sin\varphi_1 \cos\varphi_2)$$

- Use $\cos\varphi_1 = a_{11} / \|\boldsymbol{a}_1\|$, $\sin\varphi_1 = a_{12} / \|\boldsymbol{a}_1\|$, $\cos\varphi_2 = a_{21} / \|\boldsymbol{a}_2\|$, $\sin\varphi_2 = a_{22} / \|\boldsymbol{a}_2\|$

$$(\text{Area}) = \|\boldsymbol{a}_1\| \, \|\boldsymbol{a}_2\| (\sin\varphi_2 \cos\varphi_1 - \sin\varphi_1 \cos\varphi_2) = a_{11} a_{22} - a_{12} a_{21}$$

Determinant in 3D gives volume of parallelepiped

- $m = 3$



The volume is (area of base) $\times$ (height) and given as the value of the determinant

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

- The geometric interpretation of a determinant as an oriented volume is useful in establishing rules for calculation with determinants:
  - Determinant of matrix with repeated columns is zero (since two edges of the parallelepiped are identical). Example for $m=3$

$$\Delta = \begin{vmatrix} a & a & u \\ b & b & v \\ c & c & w \end{vmatrix} = abw + bcu + cav - ubc - vca - wab = 0$$

    This is more easily seen using the column notation

$$\Delta = \det(\; \boldsymbol{a}_1 \;\; \boldsymbol{a}_1 \;\; \boldsymbol{a}_3 \;\; \ldots \;) = 0$$

  - Determinant of matrix with linearly dependent columns is zero (since one edge lies in the 'hyperplane' formed by all the others)

Refer to textbook for full presentation. Most commonly used rules are:

- Separating sums in a column (similar for rows)

$$\det(\,\boldsymbol{a}_1+\boldsymbol{b}_1 \;\; \boldsymbol{a}_2 \;\; \dots \;\; \boldsymbol{a}_m\,)=\det(\,\boldsymbol{a}_1 \;\; \boldsymbol{a}_2 \;\; \dots \;\; \boldsymbol{a}_m\,)+\det(\,\boldsymbol{b}_1 \;\; \boldsymbol{a}_2 \;\; \dots \;\; \boldsymbol{a}_m\,)$$

  with $\boldsymbol{a}_i, \boldsymbol{b}_1 \in \mathbb{R}^m$

- Scalar product in a column (similar for rows)

$$\det(\,\alpha\boldsymbol{a}_1 \;\; \boldsymbol{a}_2 \;\; \dots \;\; \boldsymbol{a}_m\,)=\alpha\det(\,\boldsymbol{a}_1 \;\; \boldsymbol{a}_2 \;\; \dots \;\; \boldsymbol{a}_m\,)$$

  with $\alpha \in \mathbb{R}$

- Linear combinations of columns (similar for rows)

$$\det(\,\boldsymbol{a}_1 \;\; \boldsymbol{a}_2 \;\; \dots \;\; \boldsymbol{a}_m\,)=\det(\,\boldsymbol{a}_1 \;\; \alpha\boldsymbol{a}_1+\boldsymbol{a}_2 \;\; \dots \;\; \boldsymbol{a}_m\,)$$

  with $\alpha \in \mathbb{R}$.

- A determinant of size m can be expressed as a sum of determinants of size $m-1$ by expansion along a row or column

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1m} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{m\,m} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m2} & a_{m3} & \cdots & a_{m\,m} \end{vmatrix} -$$

$$a_{12}\begin{vmatrix} a_{21} & a_{23} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m3} & \cdots & a_{m\,m} \end{vmatrix} +$$

$$a_{13}\begin{vmatrix} a_{21} & a_{22} & a_{24} & \cdots & a_{2m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m4} & \cdots & a_{m\,m} \end{vmatrix} -$$

$$\cdots$$

$$+(-1)^{m+1}a_{1m}\begin{vmatrix} a_{21} & a_{23} & \cdots & a_{2,m-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m3} & \cdots & a_{m\,,m-1} \end{vmatrix}$$

- The formal definition of a determinant

$$\det A = \sum_{\sigma \in \Sigma} \nu(\sigma) a_{1i_1} a_{2i_2} \ldots a_{m i_m}$$

requires $mm!$ operations, a number that rapidly increases with m

- A more economical determinant is to use row and column combinations to create zeros and then reduce the size of the determinant, an algorithm reminiscent of Gauss elimination for systems

Example:

$$\begin{vmatrix} 1 & 2 & 3 \\ -1 & 0 & 1 \\ -2 & -1 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 3 & 10 \end{vmatrix} = \begin{vmatrix} 2 & 4 \\ 3 & 10 \end{vmatrix} = 20 - 12 = 8$$

The first equality comes from linear combinations of rows, i.e. row 1 is added to row 2, and row 1 multiplied by 2 is added to row 3. These linear combinations maintain the value of the determinant. The second equality comes from expansion along the first column