- Review of main linear algebra problems and their solution by a particular matrix factorization
- The eigenvalue problem
- Characteristic polynomial
- Eigenproblem in matrix form
- Eigendecomposition applications

- Compute the coordinates in a new basis, also known as *solving a linear system* $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{I}\boldsymbol{b}$ when $\boldsymbol{A} \in \mathbb{R}^{m \times m}$ square, non-singular
  1. Compute $\boldsymbol{L}\boldsymbol{U}$ factorization, $\boldsymbol{L}\boldsymbol{U} = \boldsymbol{A}$
  2. Solve $\boldsymbol{L}\boldsymbol{y} = \boldsymbol{b}$ by forward substitution
  3. Solve $\boldsymbol{U}\boldsymbol{x} = \boldsymbol{y}$ by backward substition
- Compute the closest approximation to a high dimensional vector $\boldsymbol{b} \in \mathbb{R}^m$ by linear combination of n vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n \in \mathbb{R}^m$, $\boldsymbol{A} = (\begin{array}{ccc} \boldsymbol{a}_1 & \ldots & \boldsymbol{a}_n \end{array})$, also known as the *least squares problem*
  1. Compute $\boldsymbol{Q}\boldsymbol{R}$ factorization, $\boldsymbol{Q}\boldsymbol{R} = \boldsymbol{A}$. The projector onto $C(\boldsymbol{A}) = C(\boldsymbol{Q})$ is $\boldsymbol{P}_{\boldsymbol{Q}} = \boldsymbol{Q}\boldsymbol{Q}^T$
  2. Projection of $\boldsymbol{b}$ onto $C(\boldsymbol{A})$ is $\boldsymbol{Q}\boldsymbol{Q}^T\boldsymbol{b}$. Set this equal to a linear combination of columns of $\boldsymbol{A}$, $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{Q}\boldsymbol{Q}^T\boldsymbol{b}$, Since $\boldsymbol{A} = \boldsymbol{Q}\boldsymbol{R}$, solve the triangular system $\boldsymbol{R}\boldsymbol{x} = \boldsymbol{Q}^T\boldsymbol{b}$ to find $\boldsymbol{x}$
- For a square matrix $\boldsymbol{A} \in \mathbb{C}^{m \times m}$ find those non-zero vectors whose directions are not changed by multiplication by $\boldsymbol{A}$, $\boldsymbol{A}\boldsymbol{x} = \lambda\boldsymbol{x}$, known as the *eigenvalue problem*.

Characteristic polynomial of a matrix

- Consider the eigenproblem $\boldsymbol{A}\boldsymbol{x} = \lambda\boldsymbol{x}$ for $\boldsymbol{A} \in \mathbb{C}^{m\times m}$ with $\lambda \in \mathbb{C}$, and $\boldsymbol{x} \in \mathbb{C}^m$. Rewrite as

$$\boldsymbol{A}\boldsymbol{x} = \lambda\boldsymbol{x} \Rightarrow (\boldsymbol{A} - \lambda\boldsymbol{I})\boldsymbol{x} = \boldsymbol{0}.$$

  Since $\boldsymbol{x} \neq \boldsymbol{0}$, a solution to eigenproblem exists only if $\boldsymbol{A} - \lambda\boldsymbol{I}$ is singular or $\det(\boldsymbol{A} - \lambda\boldsymbol{I}) = 0$

- Investigate form of $\det(\boldsymbol{A} - \lambda\boldsymbol{I}) = 0$

$$\det(\boldsymbol{A} - \lambda\boldsymbol{I}) = \begin{vmatrix} a_{11} - \lambda & a_{12} & a_{13} & \dots & a_{1m} \\ a_{21} & a_{22} - \lambda & a_{23} & \dots & a_{2m} \\ a_{31} & a_{32} & a_{33} - \lambda & \dots & a_{3m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mm} - \lambda \end{vmatrix}$$

- Recall algebraic definition of a determinant as sum of products of index permutations to deduce that $\det(\boldsymbol{A} - \lambda\boldsymbol{I}) = 0$ is an m^{th} degree polynomial in λ, defined as the characteristic polynomial of $\boldsymbol{A} \in \mathbb{C}^{m\times m}$

$$p_m(\lambda) = \det(\boldsymbol{A} - \lambda\boldsymbol{I})$$

Eigenproblem in matrix form

- For $\boldsymbol{A} \in \mathbb{R}^{m \times m}$, the eigenvalue problem $\boldsymbol{A}\boldsymbol{x} = \lambda \boldsymbol{x}$ ($\boldsymbol{x} \neq \boldsymbol{0}$) can be written in matrix form as

$$\boldsymbol{A}\boldsymbol{X} = \boldsymbol{X}\boldsymbol{\Lambda}, \boldsymbol{X} = \left(\begin{array}{ccc} \boldsymbol{x}_1 & \ldots & \boldsymbol{x}_m \end{array} \right) \text{eigenvector}, \boldsymbol{\Lambda} = \text{diag}(\lambda_1, \ldots, \lambda_m) \text{ eigenvalue matrices}$$

- If the column vectors of $\boldsymbol{X}$ are linearly independent, then $\boldsymbol{X}$ is invertible and $\boldsymbol{A}$ can be represented as

$$\boldsymbol{A} = \boldsymbol{X}\boldsymbol{\Lambda}\boldsymbol{X}^{-1}$$

- The above form can also be used to reduce $\boldsymbol{A}$ to diagonal form

$$\boldsymbol{\Lambda} = \boldsymbol{X}^{-1}\boldsymbol{A}\boldsymbol{X}$$

- Link to determinants: recall "determinant of product = product of determinants"

$$\det(\boldsymbol{A}\boldsymbol{X}) = \det(\boldsymbol{X}\boldsymbol{\Lambda}) \Rightarrow \det(\boldsymbol{A}) = \det(\boldsymbol{\Lambda}) \ (\text{for} \det(X) \neq 0)$$

- Diagonal forms are useful in solving linear ODE systems

$$\boldsymbol{y}' = \boldsymbol{A}\boldsymbol{y} \Leftrightarrow (\boldsymbol{X}^{-1}\boldsymbol{y}) = \boldsymbol{\Lambda}\,(\boldsymbol{X}^{-1}\boldsymbol{y})$$

- Also useful in repeatedly applying $\boldsymbol{A}$

$$\boldsymbol{u}_k = \boldsymbol{A}^k\boldsymbol{u}_0 = \boldsymbol{A}\,\boldsymbol{A}\ldots\boldsymbol{A}\,\boldsymbol{u}_0 = (\boldsymbol{X}\,\boldsymbol{\Lambda}\,\boldsymbol{X}^{-1})(\boldsymbol{X}\,\boldsymbol{\Lambda}\,\boldsymbol{X}^{-1})\ldots(\boldsymbol{X}\,\boldsymbol{\Lambda}\,\boldsymbol{X}^{-1})\boldsymbol{u}_0 = \boldsymbol{X}\,\boldsymbol{\Lambda}^k\boldsymbol{X}^{-1}\boldsymbol{u}_0$$