- Algebraic, geometric multiplicities
- Eigenvalues as roots of characteristic polynomial
- Eigenvectors as basis vectors for $N(\boldsymbol{A} - \lambda \boldsymbol{I})$
- Computation in Matlab/Octave

Definition 1. *The algebraic multiplicity of an eigenvalue λ is the number of times it appears as a repeated root of the characteristic polynomial $p(\lambda) = \det(A - \lambda I)$*

Example. $p(\lambda) = \lambda(\lambda-1)(\lambda-2)^2$ has two single roots $\lambda_1 = 0$, $\lambda_2 = 1$ and a repeated root $\lambda_{3,4} = 2$. The eigenvalue $\lambda = 2$ has an algebraic multiplicity of 2

Definition 2. *The geometric multiplicity of an eigenvalue λ is the dimension of the null space of $A - \lambda I$*

Definition 3. *An eigenvalue for which the geometric multiplicity is less than the algebraic multiplicity is said to be defective*

Theorem. *A matrix is diagonalizable if the geometric multiplicity of each eigenvalue is equal to the algebraic multiplicity of that eigenvalue.*

Eigenvalues as roots of characteristic polynomial

- Finding eigenvalues as roots of the characteristic polynomial $p(\lambda) = \det(A - \lambda I)$ is suitable for small matrices $\boldsymbol{A} \in \mathbb{R}^{m \times m}$.
  - analytical root-finding formulas are available only for $m \leqslant 4$
  - small errors in characteristic polynomial coefficients can lead to large errors in roots
- Octave/Matlab procedures to find characteristic polynomial
  - `poly(A)` function returns the coefficients
  - `roots(p)` function computes roots of the polynomial

```
octave> A=[5 -4 2; 5 -4 1; -2 2 -3]; disp(A);
```

```
  5  -4   2
  5  -4   1
 -2   2  -3
```

```
octave> p=poly(A); disp(p);
```

```
   1.00000   2.00000  -1.00000  -2.00000
```

```
octave> r=roots(p); disp(r');
```

```
   1.0000  -2.0000  -1.0000
```

```
octave>
```

- Find eigenvectors as non-trivial solutions of system $(\boldsymbol{A}-\lambda\boldsymbol{I})\boldsymbol{x}=0$

$$\lambda_1=1\Rightarrow\boldsymbol{A}-\lambda_1\boldsymbol{I}=\begin{pmatrix}4&-4&2\\5&-5&1\\-2&2&-4\end{pmatrix}\sim\begin{pmatrix}-2&2&-4\\0&0&-6\\5&-5&1\end{pmatrix}\sim\begin{pmatrix}-2&2&-4\\0&0&-6\\0&0&0\end{pmatrix}$$

  Note convenient choice of row operations to reduce amount of arithmetic, and use of knowledge that $\boldsymbol{A}-\lambda_1\boldsymbol{I}$ is singular to deduce that last row must be null

- In traditional form the above row-echelon reduced system corresponds to

$$\begin{cases}-2x_1+2x_2-4x_3&=&0\\0x_1+0x_2-6x_3&=&0\\0x_1+0x_2+0x_3&=&0\end{cases}\Rightarrow\boldsymbol{x}=\alpha\begin{pmatrix}1\\1\\0\end{pmatrix},\ \|\boldsymbol{x}\|=1\Rightarrow\alpha=1/\sqrt{2}$$

- In Octave/Matlab the computations are carried out by the null function

```
octave> null(A-eye(3))'
```

(0.70711 0.70711 0)

Ill-conditioning example

- The eigenvalues of $\boldsymbol{I} \in \mathbb{R}^{3\times 3}$ are $\lambda_{1,2,3} = 1$, but small errors in numerical computation can give roots of the characteristic polynomial with imaginary parts

```
octave> lambda=roots(poly(eye(3))); disp(lambda')
```

```
1.00001 - 0.00001i   1.00001 + 0.00001i   0.99999 - 0.00000i
```

```
octave>
```

- In the following example notice that if we slightly perturb $\boldsymbol{A}$ (by a quantity less than 0.0005=0.05%), the eigenvalues get perturb by a larger amount, e.g. 0.13%.

```
octave> A=[-2 1 -1; 5 -3 6; 5 -1 4]; disp([eig(A) eig(A+0.001*(rand(3,3)-0.5))])
```

```
-2.0000 + 0.0000i   3.0000 + 0.0000i
 3.0000 + 0.0000i  -1.9998 + 0.0189i
-2.0000 + 0.0000i  -1.9998 - 0.0189i
```

```
octave>
```

- Extracting eigenvalues and eigenvectors is a commonly encountered operation, and specialized functions exist to carry this out, including the eig function

```
octave>  [X,L]=eig(A); disp([L X]);
```

```
 -2.00000   0.00000   0.00000  -0.57735  -0.00000   0.57735
  0.00000   3.00000   0.00000   0.57735   0.70711  -0.57735
  0.00000   0.00000  -2.00000   0.57735   0.70711  -0.57735
```

```
octave> disp(null(A-3*eye(3)))
```

```
  0.00000
  0.70711
  0.70711
```

```
octave> disp(null(A+2*eye(3)))
```

```
  0.57735
 -0.57735
 -0.57735
```

```
octave>
```

- Recall definitions of eigenvalue algebraic m_λ and geometric multiplicities n_λ.

Definition. *A matrix which has $n_\lambda < m_\lambda$ for any of its eigenvalues is said to be defective.*

```
octave> A=[-2 1 -1; 5 -3 6; 5 -1 4]; [X,L]=eig(A); disp(L);
```

Diagonal Matrix

```
  -2.0000        0        0
        0   3.0000        0
        0        0  -2.0000
```

```
octave> disp(X);
```

```
  -5.7735e-01  -1.9153e-17   5.7735e-01
   5.7735e-01   7.0711e-01  -5.7735e-01
   5.7735e-01   7.0711e-01  -5.7735e-01
```

```
octave> disp(null(A+2*eye(3)));
```

```
   0.57735
  -0.57735
  -0.57735
```

```
octave> disp(rank(X))
```

```
 2
```

```
octave>
```