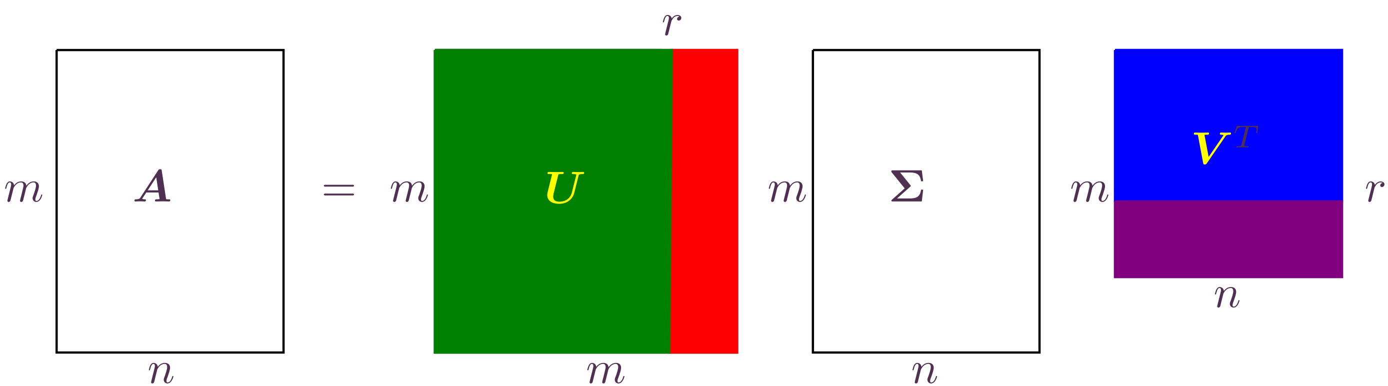
- The Singular Value Decomposition:
  - Encapsulates all aspects of a matrix (hence of a linear transformation)
  - Can be used to solve the basic problems of linear algebra
  - Is widely used in data reduction

- Singular value decomposition (SVD), for any $\boldsymbol{A} \in \mathbb{R}^{m \times n}$, $\boldsymbol{A} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^T$, with $\boldsymbol{U} \in \mathbb{R}^{m \times m}$, $\boldsymbol{V} \in \mathbb{R}^{n \times n}$ orthogonal, $\boldsymbol{\Sigma} \in \mathbb{R}_+^{m \times n}$ diagonal
- The SVD is determined by eigendecomposition of $\boldsymbol{A}^T\boldsymbol{A}$, and $\boldsymbol{A}\boldsymbol{A}^T$
  - $\boldsymbol{A}^T\boldsymbol{A} = (\boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^T)^T(\boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^T) = \boldsymbol{V}(\boldsymbol{\Sigma}^T\boldsymbol{\Sigma})\boldsymbol{V}^T$, an eigendecomposition of $\boldsymbol{A}^T\boldsymbol{A}$. The columns of $\boldsymbol{V}$ are eigenvectors of $\boldsymbol{A}^T\boldsymbol{A}$ and called *right singular vectors* of $\boldsymbol{A}$
  - $\boldsymbol{A}\boldsymbol{A}^T = (\boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^T)(\boldsymbol{U}\boldsymbol{\Sigma}^T\boldsymbol{V}^T)^T = \boldsymbol{U}(\boldsymbol{\Sigma}\boldsymbol{\Sigma}^T)\boldsymbol{U}^T$, an eigendecomposition of $\boldsymbol{A}\boldsymbol{A}^T$. The columns of $\boldsymbol{U}$ are eigenvectors of $\boldsymbol{A}\boldsymbol{A}^T$ and called *left singular vectors* of $\boldsymbol{A}$
  - The matrix $\boldsymbol{\Sigma}$ has zero elements except for the diagonal that contains σ_i, the *singular values* of $\boldsymbol{A}$ computed as the square roots of the eigenvalues of $\boldsymbol{A}^T\boldsymbol{A}$ (or $\boldsymbol{A}\boldsymbol{A}^T$)

$$\boldsymbol{\Sigma} = \begin{pmatrix} \sigma_1 & & & & & \\ & \sigma_2 & & & & \\ & & \ddots & & & \\ & & & \sigma_r & & \\ & & & & 0 & \\ & & & & & \ddots \end{pmatrix} \in \mathbb{R}_+^{m \times n}$$

SVD and matrix subspaces

- The SVD of $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ reveals: $\operatorname{rank}(\boldsymbol{A})$, bases for $C(\boldsymbol{A}), N(\boldsymbol{A}^T), C(\boldsymbol{A}^T), N(\boldsymbol{A})$



$$\boldsymbol{A} = \begin{pmatrix} \boldsymbol{u}_1 & \dots & \boldsymbol{u}_r & \boldsymbol{u}_{r+1} & \dots & \boldsymbol{u}_m \end{pmatrix} \begin{pmatrix} \sigma_1 & & & & \\ & \ddots & & & \\ & & \sigma_r & & \\ & & & 0 & \\ & & & & \ddots \end{pmatrix} \begin{pmatrix} \boldsymbol{v}_1{}^T \\ \vdots \\ \boldsymbol{v}_r{}^T \\ \boldsymbol{v}_{r+1}{}^T \\ \vdots \\ \boldsymbol{v}_n{}^T \end{pmatrix}$$

SVD solution of linear system problems

- Change of coordinates (solving a linear system) $\boldsymbol{A}\boldsymbol{x}=\boldsymbol{b}$, $\boldsymbol{A}\in\mathbb{R}^{m\times m}$, $\boldsymbol{x},\boldsymbol{b}\in\mathbb{R}^m$

$\boldsymbol{LU}$ solution	Operations (flops)	SVD solution	Operations (flops)
Factor $\boldsymbol{LU}=\boldsymbol{A}$	$\mathcal{O}\left(\frac{2m^3}{3}\right)$	Factor $\boldsymbol{U\Sigma V}^T=\boldsymbol{A}$	$\mathcal{O}(11m^3)$
Forward substitution $\boldsymbol{Ly}=\boldsymbol{b}$	$\mathcal{O}\left(\frac{m^2}{2}\right)$	Compute $\boldsymbol{c}=\boldsymbol{U}^T\boldsymbol{b}$	$\mathcal{O}(m^2)$
Back substitute $\boldsymbol{Ux}=\boldsymbol{y}$	$\mathcal{O}\left(\frac{m^2}{2}\right)$	Compute $\boldsymbol{d}=\boldsymbol{\Sigma}^{-1}\boldsymbol{c}$	$\mathcal{O}(m)$
		$d_i=c_i/\sigma_i$, $i=1,\ldots,m$	
		Compute $\boldsymbol{x}=\boldsymbol{Vd}$	$\mathcal{O}(m^2)$

flop = floating point operation, defined as an addition and multiplication

- SVD can be used for solving a linear system, but is generally more expensive than the standard $\boldsymbol{LU}$ factorization (Gaussian elimination)

- Find closest approximation to $\boldsymbol{b} \in \mathbb{R}^m$ by linear combination of column vectors of $\boldsymbol{A} \in \mathbb{R}^{m \times n}$, $m > n$, $\min_{\boldsymbol{x}} \|\boldsymbol{b} - \boldsymbol{A}\boldsymbol{x}\|$

$\boldsymbol{QR}$ solution	Operations (flops)	SVD solution	Operations (flops)
Factor $\boldsymbol{QR} = \boldsymbol{A}$	$\mathcal{O}\left(2mn^2 - \frac{2n^3}{3}\right)$	Factor $\boldsymbol{U\Sigma V}^T = \boldsymbol{A}$	$\mathcal{O}(2mn^2 + 11n^3)$
Compute $\boldsymbol{c} = \boldsymbol{Q}^T\boldsymbol{b}$	$\mathcal{O}(mn)$	Compute $\boldsymbol{c} = \boldsymbol{U}^T\boldsymbol{b}$	$\mathcal{O}(mn)$
Back substitute $\boldsymbol{Rx} = \boldsymbol{c}$	$\mathcal{O}\left(\frac{n^2}{2}\right)$	Compute $\boldsymbol{d} = \boldsymbol{\Sigma}^+\boldsymbol{c}$	$\mathcal{O}(n)$
		$d_i = c_i / \sigma_i,\ i = 1, \ldots, n$	
		Compute $\boldsymbol{x} = \boldsymbol{Vd}$	$\mathcal{O}(n^2)$

flop = floating point operation, defined as an addition and multiplication

- SVD can be used for solving a least squares problem, but is generally more expensive than the standard $\boldsymbol{QR}$ factorization (Gram-Schmidt orthogonalization)