

- New concepts:
 - Gaussian multiplier matrix
 - LU factorization

- Recall the basic operation in row echelon reduction: constructing a linear combination of rows to form zeros beneath the main diagonal, e.g.

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ a_{31} & a_{32} & \dots & a_{3m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mm} \end{pmatrix} \sim \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ 0 & a_{22} - \frac{a_{21}}{a_{11}}a_{12} & \dots & a_{2m} - \frac{a_{21}}{a_{11}}a_{1m} \\ 0 & a_{32} - \frac{a_{31}}{a_{11}}a_{12} & \dots & a_{3m} - \frac{a_{31}}{a_{11}}a_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a_{m2} - \frac{a_{m1}}{a_{11}}a_{12} & \dots & a_{mm} - \frac{a_{m1}}{a_{11}}a_{1m} \end{pmatrix}$$

- This can be stated as a matrix multiplication operation, with $l_{i1} = a_{i1}/a_{11}$

$$\begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ -l_{21} & 1 & 0 & \dots & 0 \\ -l_{31} & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -l_{m1} & 0 & 0 & \dots & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ a_{31} & a_{32} & \dots & a_{3m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mm} \end{pmatrix} =$$

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ 0 & a_{22} - l_{21}a_{12} & \dots & a_{2m} - l_{21}a_{1m} \\ 0 & a_{32} - l_{31}a_{12} & \dots & a_{3m} - l_{31}a_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a_{m2} - l_{m1}a_{12} & \dots & a_{mm} - l_{m1}a_{1m} \end{pmatrix}$$

Definition. *The matrix*

$$\mathbf{L}_k = \begin{pmatrix} 1 & \dots & 0 & \dots & 1 \\ 0 & \ddots & 0 & \dots & 0 \\ 0 & \dots & 1 & \dots & 0 \\ 0 & \dots & -l_{k+1,k} & \dots & 0 \\ 0 & \dots & -l_{k+2,k} & \dots & 0 \\ \vdots & \dots & \vdots & \ddots & \vdots \\ 0 & \dots & -l_{m,k} & \dots & 1 \end{pmatrix}$$

with $l_{i,k} = a_{i,k}^{(k)} / a_{k,k}^{(k)}$, and $\mathbf{A}^{(k)} = \left(a_{i,j}^{(k)} \right)$ the matrix obtained after step k of row echelon reduction (or, equivalently, Gaussian elimination) is called a Gaussian **multiplier matrix**.

- For $\mathbf{A} \in \mathbb{R}^{m \times m}$ nonsingular, the successive steps in row echelon reduction (or Gaussian elimination) correspond to successive multiplications on the left by Gaussian multiplier matrices

$$\mathbf{L}_{m-1}\mathbf{L}_{m-2}\dots\mathbf{L}_2\mathbf{L}_1\mathbf{A} = \mathbf{U}$$

- The inverse of a Gaussian multiplier is

$$\mathbf{L}_k^{-1} = \begin{pmatrix} 1 & \dots & 0 & \dots & 1 \\ 0 & \ddots & 0 & \dots & 0 \\ 0 & \dots & 1 & \dots & 0 \\ 0 & \dots & l_{k+1,k} & \dots & 0 \\ 0 & \dots & l_{k+2,k} & \dots & 0 \\ \vdots & \dots & \vdots & \ddots & \vdots \\ 0 & \dots & l_{m,k} & \dots & 1 \end{pmatrix} = \mathbf{I} - (\mathbf{L}_k - \mathbf{I})$$

- From $(\mathbf{L}_{m-1}\mathbf{L}_{m-2}\dots\mathbf{L}_2\mathbf{L}_1)\mathbf{A}=\mathbf{U}$ obtain

$$\mathbf{A}=(\mathbf{L}_{m-1}\mathbf{L}_{m-2}\dots\mathbf{L}_2\mathbf{L}_1)^{-1}\mathbf{U}=\mathbf{L}_1^{-1}\mathbf{L}_2^{-1}\cdot\dots\cdot\mathbf{L}_{m-1}^{-1}\mathbf{U}=\mathbf{L}\mathbf{U}$$

- Due to the simple form of \mathbf{L}_k^{-1} the matrix \mathbf{L} is easily obtained as

$$\mathbf{L}=\begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ l_{2,1} & 1 & 0 & \dots & 0 & 0 \\ l_{3,1} & l_{3,2} & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ l_{m-1,1} & l_{m-1,2} & l_{m-1,3} & \dots & 1 & 0 \\ l_{m,1} & l_{m,2} & l_{m,3} & \dots & l_{m,m-1} & 1 \end{pmatrix}$$

- Using the concept of an LU factorization, finding the solution to a linear system $Ax = b$ can be formulated as the following steps:
 1. Find the LU factorization, $LU = A$
 2. Replace factorization into system and regroup $Ax = b \Leftrightarrow (LU)x = b \Leftrightarrow Ly = b$. Solve the lower triangular $Ly = b$ system to find y
 3. Solve the upper triangular system $Ux = y$ to find x
- The above formulation of the steps within Gaussian elimination is very useful in computer calculations