- Algebraic definition of a determinant using permutations
- Applications of determinants: volumes of simplicia

Algebraic definition of a determinant through permutations

- Denote a permutation by

$$\sigma = \begin{pmatrix} 1 & 2 & \ldots & m \\ i_1 & i_2 & \cdots & i_m \end{pmatrix}$$

  with $i_1, \ldots, i_m \in \{1, \ldots, m\}$, $i_j \neq i_k$ for $j \neq k$

- The sign of a permutation, $\nu(\sigma)$ is the number of pair swaps needed to obtain the permutation starting from the identity permutation

$$\begin{pmatrix} 1 & 2 & \ldots & m \\ 1 & 2 & \cdots & m \end{pmatrix}$$

- Let Σ be the set of all possible permutations ($m!$ of these are possible)
- The algebraic definition of a determinant is

$$\det A = \sum_{\sigma \in \Sigma} \nu(\sigma) a_{1i_1} a_{2i_2} \ldots a_{m\,i_m}.$$

- The formal definition of a determinant

$$\det A = \sum_{\sigma\in\Sigma} \nu(\sigma) a_{1i_1} a_{2i_2} \ldots a_{m\,i_m}$$

requires $mm!$ operations, a number that rapidly increases with m

- A more economical determinant is to use row and column combinations to create zeros and then reduce the size of the determinant, an algorithm reminiscent of Gauss elimination for systems

Example:

$$\begin{vmatrix} 1 & 2 & 3 \\ -1 & 0 & 1 \\ -2 & -1 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 3 & 10 \end{vmatrix} = \begin{vmatrix} 2 & 4 \\ 3 & 10 \end{vmatrix} = 20 - 12 = 8$$

The first equality comes from linear combinations of rows, i.e. row 1 is added to row 2, and row 1 multiplied by 2 is added to row 3. These linear combinations maintain the value of the determinant. The second equality comes from expansion along the first column

1. $d=1$, define an interval by $\{\boldsymbol{r}_1, \boldsymbol{r}_2\}$, with $\boldsymbol{r}_1, \boldsymbol{r}_2 \in \mathbb{R}$, $\boldsymbol{r}_1 = (x_1)$, $\boldsymbol{r}_2 = (x_2)$. The signed length of the interval is $l = x_2 - x_1$ and can be computed as

$$\Delta = \begin{vmatrix} 1 & 1 \\ x_1 & x_2 \end{vmatrix}$$

2. $d=2$, define a triangle by $\{\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3\}$, with $\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3 \in \mathbb{R}^2$. The signed area is

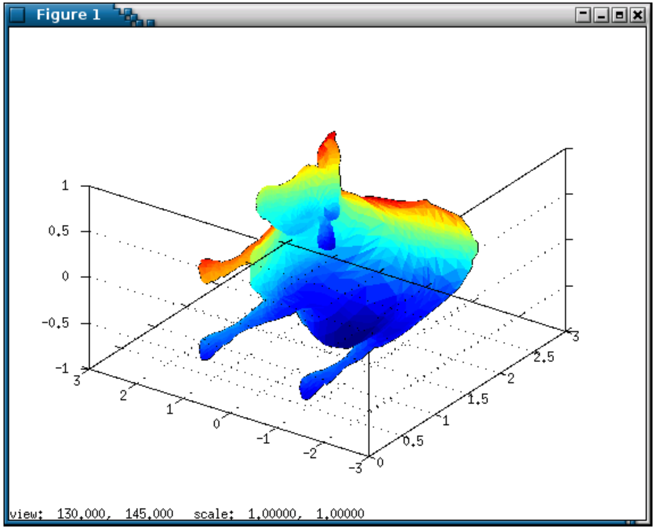
$$\Delta = \frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$$

3. $d=3$, define a tetrahedron by $\{\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3, \boldsymbol{r}_4\}$. The signed volume is

$$\Delta = \frac{1}{6}\begin{vmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{vmatrix}$$

Example: How big is the hide on that cow?

```
octave> cd ~/courses/MATH547/lessons/ply;
octave> [x y z]=textread('cow.nodes','%f %f %f');
octave> [np p1 p2 p3]=textread('cow.faces','%d %d %d %d');
octave> tri=[p1+1 p2+1 p3+1];
octave> trisurf(tri,x,y,z); print -mono -deps cow.eps;
```



How big is that cow? Calculate area of a triangle and of all triangles

```
octave> disp(tri(1,:));
```

1 2 3

```
octave> 0.5*abs(det([1 1 1; x(tri(1,1)) x(tri(1,2)) x(tri(1,3)); y(tri(1,1)) y(tri(1,2))
        y(tri(1,3))]))
```

0.0018142

```
octave> Ahide=0.;
```

```
octave> for n=1:max(size(tri))
          Ahide = Ahide + 0.5*abs(det([1 1 1; x(tri(n,1)) x(tri(n,2)) x(tri(n,3));
        y(tri(n,1)) y(tri(n,2)) y(tri(n,3))]));
        end;
```

```
octave> disp(Ahide);
```

13.265

```
octave>
```