Lesson 20: Applications of determinants: cross products, Cramer's rule

- Cramer's rule for solving linear systems
- Cross products

Deducing Cramer's rule for solving linear systems

- Determinants give an explicit expression for the solution to a linear system

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$$

$\boldsymbol{A} \in \mathbb{R}^{m \times m}, \boldsymbol{x}, \boldsymbol{b} \in \mathbb{R}^m$, $\boldsymbol{A}$ non-singular with columns $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m \in \mathbb{R}^m$

- Note that the solution can be expressed as $\boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{b}$, and the inverse has the property

$$\boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{A}^{-1}\begin{pmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \ldots & \boldsymbol{a}_m \end{pmatrix} = \boldsymbol{I} = \begin{pmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \ldots & \boldsymbol{e}_m \end{pmatrix}$$

- Consider now the effect of multiplying $\boldsymbol{A}^{-1}$ with a modification of the matrix $\boldsymbol{A}$ in which the first column is replaced by $\boldsymbol{b}$

$$\boldsymbol{A}^{-1}\begin{pmatrix} \boldsymbol{b} & \boldsymbol{a}_2 & \ldots & \boldsymbol{a}_m \end{pmatrix} = \begin{pmatrix} \boldsymbol{x} & \boldsymbol{e}_2 & \ldots & \boldsymbol{e}_m \end{pmatrix}$$

The determinant of the resulting product is

$$\Delta_1 = \det\begin{pmatrix} \boldsymbol{x} & \boldsymbol{e}_2 & \ldots & \boldsymbol{e}_m \end{pmatrix} = \begin{vmatrix} x_1 & 0 & & \\ x_2 & 1 & & \\ \vdots & \ldots & \ddots & \\ x_m & 0 & & 1 \end{vmatrix} = x_1$$

Cramer's rule

- When replacing column i in A by b we obtain a matrix whose determinant is x_i
- Cramer's rule sues the above facts to express the solution components as

$$x_i = \frac{\Delta_i}{\Delta} \tag{1}$$

  with Δ the determinant of $\boldsymbol{A}$, and Δ_i the determinant of the matrix obtained by replacing column i of $\boldsymbol{A}$ by $\boldsymbol{b}$

- Cramer's rule involves computation of $m+1$ determinants. Since the efficient way of computing each determinant is to use row operations to obtain reduction to a triangular form, applying Cramer's rule is roughly equivalent to $m+1$ triangularizations (as in Gaussian elimination)
- From the above we see that though (1) is a closed-form formula, it is not practical for numerical evaluations since it involves more computation than Gaussian elimination
- Cramer's rule is however useful in various analytical evaluations of the solution to a linear system

- Consider $u, v \in \mathbb{R}^3$. We've introduced the idea of a scalar product

$$u \cdot v = u^T v = u_1 v_1 + u_2 v_2 + u_3 v_3$$

in which from two vectors one obtains a scalar

- We've also introduced the idea of an exterior product

$$u v^T = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix} = \begin{pmatrix} u_1 v_1 & u_1 v_2 & u_1 v_3 \\ u_2 v_1 & u_2 v_2 & u_2 v_3 \\ u_3 v_1 & u_3 v_2 & u_3 v_3 \end{pmatrix}$$

in which a matrix is obtained from two vectors

- Another product of two vectors is also useful, the cross product, most conveniently expressed in determinant-like form

$$u \times v = \begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = (u_2 v_3 - v_2 u_3)\boldsymbol{e}_1 + (u_3 v_1 - v_3 u_1)\boldsymbol{e}_2 + (u_1 v_2 - v_1 u_2)\boldsymbol{e}_3$$

Cross product applications

- The cross product often arises in physics. An example is the computation of a rotational velocity as

$$v = \omega \times r$$

  with ω the rotation vector, and r the position vector from the center of rotation to the point at which the velocity v is computed

- The cross product vector is orthogonal to the factor vectors

$$u \perp u \times v, v \perp u \times v$$

- Verify by computing the mixed product

$$(u \times v) \cdot u = (u_2 v_3 - v_2 u_3) u_1 + (u_3 v_1 - v_3 u_1) u_2 + (u_1 v_2 - v_1 u_2) u_3 = 0$$