- Review of main linear algebra problems and their solution by a particular matrix factorization
- The eigenvalue problem

- Compute the coordinates in a new basis, also known as *solving a linear system* $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{I}\boldsymbol{b}$ when $\boldsymbol{A} \in \mathbb{R}^{m \times m}$ square, non-singular

  1. Compute $\boldsymbol{L}\boldsymbol{U}$ factorization, $\boldsymbol{L}\boldsymbol{U} = \boldsymbol{A}$
  2. Solve $\boldsymbol{L}\boldsymbol{y} = \boldsymbol{b}$ by forward substitution
  3. Solve $\boldsymbol{U}\boldsymbol{x} = \boldsymbol{y}$ by backward substition

- Compute the closest approximation to a high dimensional vector $\boldsymbol{b} \in \mathbb{R}^m$ by linear combination of n vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n \in \mathbb{R}^m$, $\boldsymbol{A} = (\begin{array}{ccc} \boldsymbol{a}_1 & \ldots & \boldsymbol{a}_n \end{array})$, also known as the *least squares problem*

  1. Compute $\boldsymbol{Q}\boldsymbol{R}$ factorization, $\boldsymbol{Q}\boldsymbol{R} = \boldsymbol{A}$. The projector onto $C(\boldsymbol{A}) = C(\boldsymbol{Q})$ is $\boldsymbol{P}_{\boldsymbol{Q}} = \boldsymbol{Q}\boldsymbol{Q}^T$
  2. Projection of $\boldsymbol{b}$ onto $C(\boldsymbol{A})$ is $\boldsymbol{Q}\boldsymbol{Q}^T\boldsymbol{b}$. Set this equal to a linear combination of columns of $\boldsymbol{A}$, $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{Q}\boldsymbol{Q}^T\boldsymbol{b}$, Since $\boldsymbol{A} = \boldsymbol{Q}\boldsymbol{R}$, solve the triangular system $\boldsymbol{R}\boldsymbol{x} = \boldsymbol{Q}^T\boldsymbol{b}$ to find $\boldsymbol{x}$

- For a square matrix $\boldsymbol{A} \in \mathbb{C}^{m \times m}$ find those non-zero vectors whose directions are not changed by multiplication by $\boldsymbol{A}$, $\boldsymbol{A}\boldsymbol{x} = \lambda\boldsymbol{x}$, known as the *eigenvalue problem*.

- Consider the eigenproblem $\boldsymbol{A}\boldsymbol{x} = \lambda\boldsymbol{x}$ for $\boldsymbol{A} \in \mathbb{C}^{m\times m}$ with $\lambda \in \mathbb{C}$, and $\boldsymbol{x} \in \mathbb{C}^m$. Rewrite as

$$\boldsymbol{A}\boldsymbol{x} = \lambda\boldsymbol{x} \Rightarrow (\boldsymbol{A} - \lambda\boldsymbol{I})\boldsymbol{x} = \boldsymbol{0}.$$

  Since $\boldsymbol{x} \neq \boldsymbol{0}$, deduce that a solution to the eigenproblem exists only if $\boldsymbol{A} - \lambda\boldsymbol{I}$ is singular or

$$\det(\boldsymbol{A} - \lambda\boldsymbol{I}) = 0$$

- Investigate form of $\det(\boldsymbol{A} - \lambda\boldsymbol{I}) = 0$

$$\det(\boldsymbol{A} - \lambda\boldsymbol{I}) = \begin{vmatrix} a_{11} - \lambda & a_{12} & a_{13} & \dots & a_{1m} \\ a_{21} & a_{22} - \lambda & a_{23} & \dots & a_{2m} \\ a_{31} & a_{32} & a_{33} - \lambda & \dots & a_{3m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mm} - \lambda \end{vmatrix}$$

- Recall algebraic definition of a determinant as sum of products of index permutations to deduce that $\det(\boldsymbol{A} - \lambda\boldsymbol{I}) = 0$ is an m^{th} degree polynomial in λ, defined as the characteristic polynomial of $\boldsymbol{A} \in \mathbb{C}^{m\times m}$

$$p_m(\lambda) = \det(\boldsymbol{A} - \lambda\boldsymbol{I})$$