Lesson 22: Matrix eigendecomposition, Matrix diagonalization

- Eigenproblem in matrix form
- Matrix eigendecomposition, Matrix diagonalization
- Algebraic, geometric multiplicities

- For $\boldsymbol{A} \in \mathbb{R}^{m \times m}$, the eigenvalue problem $\boldsymbol{A}\boldsymbol{x} = \lambda \boldsymbol{x}$ $(\boldsymbol{x} \neq \mathbf{0})$ can be written in matrix form as

$$\boldsymbol{A}\boldsymbol{X} = \boldsymbol{X}\boldsymbol{\Lambda}, \boldsymbol{X} = \left(\begin{array}{ccc} \boldsymbol{x}_1 & \ldots & \boldsymbol{x}_m \end{array} \right) \text{eigenvector}, \boldsymbol{\Lambda} = \operatorname{diag}(\lambda_1, \ldots, \lambda_m) \text{ eigenvalue matrices}$$

- If the column vectors of $\boldsymbol{X}$ are linearly independent, then $\boldsymbol{X}$ is invertible and $\boldsymbol{A}$ can be represented as

$$\boldsymbol{A} = \boldsymbol{X}\boldsymbol{\Lambda}\boldsymbol{X}^{-1}$$

- The above form can also be used to reduce $\boldsymbol{A}$ to diagonal form

$$\boldsymbol{\Lambda} = \boldsymbol{X}^{-1}\boldsymbol{A}\boldsymbol{X}$$

- Link to determinants: recall "determinant of product = product of determinants"

$$\det(\boldsymbol{A}\boldsymbol{X}) = \det(\boldsymbol{X}\boldsymbol{\Lambda}) \Rightarrow \det(\boldsymbol{A}) = \det(\boldsymbol{\Lambda}) \ (\text{for} \det(X) \neq 0)$$

- Diagonal forms are useful in solving linear ODE systems

$$y' = \mathrm{Ay} \Leftrightarrow (X^{-1}y) = \Lambda\,(X^{-1}y)$$

- Also useful in repeatedly applying A

$$u_k = A^k u_0 = A A \ldots A u_0 = (X \Lambda X^{-1})(X \Lambda X^{-1}) \ldots (X \Lambda X^{-1}) u_0 = X \Lambda^k X^{-1} u_0$$

Definition 1. *The algebraic multiplicity of an eigenvalue λ is the number of times it appears as a repeated root of the characteristic polynomial $p(\lambda) = \det(A - \lambda I)$*

Example. $p(\lambda) = \lambda(\lambda - 1)(\lambda - 2)^2$ has two single roots $\lambda_1 = 0$, $\lambda_2 = 1$ and a repeated root $\lambda_{3,4} = 2$. The eigenvalue $\lambda = 2$ has an algebraic multiplicity of 2

Definition 2. *The geometric multiplicity of an eigenvalue λ is the dimension of the null space of $A - \lambda I$*

Definition 3. *An eigenvalue for which the geometric multiplicity is less than the algebraic multiplicity is said to be defective*

Proposition 4. *A matrix is diagonalizable is the geometric multiplicity of each eigenvalue is equal to the algebraic multiplicity of that eigenvalue.*