- Eigenvalues as roots of characteristic polynomial
- Eigenvectors as basis vectors for $N(\boldsymbol{A} - \lambda \boldsymbol{I})$
- Computation in Matlab/Octave
- Application: study of vibrations

Eigenvalues as roots of characteristic polynomial

- Finding eigenvalues as roots of the characteristic polynomial $p(\lambda) = \det(A - \lambda I)$ is suitable for small matrices $\boldsymbol{A} \in \mathbb{R}^{m \times m}$.
  - analytical root-finding formulas are available only for $m \leqslant 4$
  - small errors in the characteristic polynomial coefficients can lead to large errors in the roots, the polynomial root-finding problem is said to be ill-conditioned
- Octave/Matlab procedures to find characteristic polynomial
  - `poly(A)` function returns the coefficients
  - `roots(p)` function computes roots of the polynomial

```
octave> A=[5 -4 2; 5 -4 1; -2 2 -3]; disp(A);
   5  -4   2
   5  -4   1
  -2   2  -3
octave> p=poly(A); disp(p);
   1.00000   2.00000  -1.00000  -2.00000
octave> r=roots(p); disp(r');
   1.0000  -2.0000  -1.0000
```

- Find eigenvectors as non-trivial solutions of system $(\boldsymbol{A}-\lambda\boldsymbol{I})\boldsymbol{x}=0$

$$\lambda_1=1\Rightarrow\boldsymbol{A}-\lambda_1\boldsymbol{I}=\left(\begin{array}{ccc}4 & -4 & 2\\ 5 & -5 & 1\\ -2 & 2 & -4\end{array}\right)\sim\left(\begin{array}{ccc}-2 & 2 & -4\\ 0 & 0 & -6\\ 5 & -5 & 1\end{array}\right)\sim\left(\begin{array}{ccc}-2 & 2 & -4\\ 0 & 0 & -6\\ 0 & 0 & 0\end{array}\right)$$

  Note convenient choice of row operations to reduce amount of arithmetic, and use of knowledge that $\boldsymbol{A}-\lambda_1\boldsymbol{I}$ is singular to deduce that last row must be null

- In traditional form the above row-echelon reduced system corresponds to

$$\left\{\begin{array}{rcl}-2x_1+2x_2-4x_3 & = & 0\\ 0x_1+0x_2-6x_3 & = & 0\\ 0x_1+0x_2+0x_3 & = & 0\end{array}\right.\Rightarrow\boldsymbol{x}=\alpha\left(\begin{array}{c}1\\ 1\\ 0\end{array}\right),\ \|\boldsymbol{x}\|=1\Rightarrow\alpha=$$

- In Octave/Matlab the computations are carried out by the null function

```
octave> null(A-eye(3))'
```

(0.70711 0.70711 0)

- The eigenvalues of $\boldsymbol{I} \in \mathbb{R}^{3\times 3}$ are $\lambda_{1,2,3} = 1$, but small errors in numerical computation can give roots of the characteristic polynomial with imaginary parts

```
octave> roots(poly(eye(3)))
```

$$\begin{pmatrix} 1 + 9.094e - 06 \cdot i \\ 1 - 9.094e - 06 \cdot i \\ 0.99999 \end{pmatrix}$$

- In the following example notice that if we slightly perturb $\boldsymbol{A}$ (by a quantity less than 0.0005=0.05%), the eigenvalues get perturb by a larger amount, e.g. 0.13%.

```
octave> [eig(A) eig(A+0.001*(rand(3,3)-0.5))]
```

$$\begin{pmatrix} 1 & 1.0013 \\ -1 & -0.99977 \\ -2 & -2.0013 \end{pmatrix}$$

```
octave>
```

- Extracting eigenvalues and eigenvectors is a commonly encountered operation, and specialized functions exist to carry this out, including the eig function

```
octave>  [X,L]=eig(A);
```

```
octave> disp(L);
```

Diagonal Matrix

```
  -2.0000        0        0
        0   3.0000        0
        0        0  -2.0000
```

```
octave> null(A-3*eye(3))
```

$$\begin{pmatrix} 0 \\ 0.70711 \\ 0.70711 \end{pmatrix}$$

```
octave> null(A+2*eye(3))
```

$$\begin{pmatrix} 0.57735 \\ -0.57735 \\ -0.57735 \end{pmatrix}$$

```
octave> null(eye(3)-eye(3))
```

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

```
octave>
```

- Recall definitions of eigenvalue algebraic m_λ and geometric multiplicities n_λ.

Definition. *A matrix which has $n_\lambda < m_\lambda$ for any of its eigenvalues is said to be defective.*

```
octave> A=[-2 1 -1; 5 -3 6; 5 -1 4]; [X,L]=eig(A); disp(L);
```

```
Diagonal Matrix

  -2.0000        0        0
        0   3.0000        0
        0        0  -2.0000
```

```
octave> disp(X);
```

```
  -5.7735e-01  -1.9153e-17   5.7735e-01
   5.7735e-01   7.0711e-01  -5.7735e-01
   5.7735e-01   7.0711e-01  -5.7735e-01
```

```
octave> disp(null(A+2*eye(3)));
```

```
   0.57735
  -0.57735
  -0.57735
```

```
octave>
```