- Review of matrix decompositions:
  - $\boldsymbol{L}\boldsymbol{U} = \boldsymbol{A}$ factorization (Gaussian elimination), used to solve linear systems (compute coordinates in new basis)
  - $\boldsymbol{Q}\boldsymbol{R} = \boldsymbol{A}$ factorization (Gram-Schmidt algorithm), used to solve least squares problems (compute best possible approximation)
  - $\boldsymbol{A}\boldsymbol{X} = \boldsymbol{X}\boldsymbol{\Lambda}$, eigenproblem. If $\boldsymbol{X}$ nonsingular, eigendecomposition $\boldsymbol{X}\boldsymbol{\Lambda}\boldsymbol{X}^{-1} = \boldsymbol{A}$ (reduction to diagonal form)
- Additional matrix decompositions:
  - $\boldsymbol{Q}\boldsymbol{T}\boldsymbol{Q}^T = \boldsymbol{A}$, Schur decomposition (reduction to triangular form)
  - $\boldsymbol{P}\boldsymbol{J}\boldsymbol{P}^{-1} = \boldsymbol{A}$, Jordan decomposition (reduction to disjoint eigenspaces)
  - $\boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^T = \boldsymbol{A}$, singular value decomposition (SVD, reduction to diagonal form, but with different bases in the domain, codomain)

Theorem. *(Schur) Any square matrix* $\boldsymbol{A} \in \mathbb{R}^{m \times m}$ *can be decomposed as* $\boldsymbol{A} = \boldsymbol{Q}\boldsymbol{T}\boldsymbol{Q}^T$*, with* $\boldsymbol{T} \in \mathbb{R}^{m \times m}$ *upper triangular* ($t_{ij} = 0$ *for* $i > j$) *and* $\boldsymbol{Q} \in \mathbb{R}^{m \times m}$ *orthogonal* ($\boldsymbol{Q}\boldsymbol{Q}^T = \boldsymbol{I}$)*.*

Proof. *By induction. Consider the eigenvalue relationship* $\boldsymbol{A}\boldsymbol{x} = \lambda \boldsymbol{x}$ *with* $\|\boldsymbol{x}\| = 1$*. Form an orthogonal matrix* $\boldsymbol{U} = \begin{pmatrix} \boldsymbol{x} & \boldsymbol{u}_2 & \dots & \boldsymbol{u}_m \end{pmatrix}$*. Then*

$$\boldsymbol{U}^T\boldsymbol{A}\boldsymbol{U} = \begin{pmatrix} \boldsymbol{x}^T \\ \boldsymbol{u}_2^T \\ \vdots \\ \boldsymbol{u}_m^T \end{pmatrix} \begin{pmatrix} \boldsymbol{A}\boldsymbol{x} & \boldsymbol{A}\boldsymbol{u}_2 & \dots & \boldsymbol{A}\boldsymbol{u}_m \end{pmatrix} = \begin{pmatrix} \boldsymbol{x}^T \\ \boldsymbol{u}_2^T \\ \vdots \\ \boldsymbol{u}_m^T \end{pmatrix} \begin{pmatrix} \lambda\boldsymbol{x} & \boldsymbol{A}\boldsymbol{u}_2 & \dots & \boldsymbol{A}\boldsymbol{u}_m \end{pmatrix} = \begin{pmatrix} \lambda & \boldsymbol{w} \\ 0 & \boldsymbol{B} \end{pmatrix}.$$

By induction hypothesis $\boldsymbol{B} = \boldsymbol{V}\boldsymbol{S}\boldsymbol{V}^T$ *with* $\boldsymbol{S}$ *triangular,* $\boldsymbol{V}$ *orthogonal so*

$$\boldsymbol{U}^T\boldsymbol{A}\boldsymbol{U} = \begin{pmatrix} \lambda & \boldsymbol{w} \\ 0 & \boldsymbol{V}\boldsymbol{S}\boldsymbol{V}^T \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \boldsymbol{V} \end{pmatrix} \begin{pmatrix} \lambda & \boldsymbol{w} \\ 0 & \boldsymbol{S} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \boldsymbol{V} \end{pmatrix}^T \Rightarrow \boldsymbol{Q} = \boldsymbol{U} \begin{pmatrix} 1 & 0 \\ 0 & \boldsymbol{V} \end{pmatrix}$$

$\square$

- A non-defective square matrix can be diagonalized $\boldsymbol{X}^{-1}\boldsymbol{A}\boldsymbol{X} = \boldsymbol{\Lambda}$
- Reduction to diagonal form has shown to be very useful, e.g., when solving ODE systems
- Recall that computation of a matrix inverse is costly in general, but simple for orthogonal matrices, $\boldsymbol{Q}^T\boldsymbol{Q} = \boldsymbol{I}$

Definition. *A matrix is unitarily diagonalizable if it admits an complete, orthonormal set of eigenvectors.*

Definition. *A matrix $\boldsymbol{A} \in \mathbb{R}^{m\times m}$ is normal if $\boldsymbol{A}^T\boldsymbol{A} = \boldsymbol{A}\boldsymbol{A}^T$, or if $\boldsymbol{A} \in \mathbb{C}^{m\times m}$, $\boldsymbol{A}^*\boldsymbol{A} = \boldsymbol{A}\boldsymbol{A}^*$.*

- Corollaries of the Schur theorem:
  - orthogonal matrices are unitarily diagonalizable
  - symmetric matrices are unitarily diagonalizable
  - skew-symmetric matrices are unitarily diagonalizable
  - normal matrices are unitarily diagonalizable