FINAL EXAMINATION

Solve the following problems (5 course points each). Present a brief motivation of your method of solution. Explicitly state any conditions that must be met for solution procedure to be valid. Organize your computation and writing so the solution you present is readily legible. No credit is awarded for statement of the final answer to a problem without presentation of solution procedure.

This is an open-book test, and you are free to consult the textbook or use software to check your solution. Your submission must however reflect your individual effort with no aid from any other person. If you studied the course material and understood solutions to the homework assignments, drafting examination question solutions in TeXmacs should take about 2 hours. The allotted time is 3:10 hours, thus also providing flexibility for internet connection interruption.

Draft your solution in TeXmacs in this file. Upload your answer into Sakai both as a TeXmacs, and pdf. Allow at least 10 minutes before the submission cut-off time to ensure you can upload your file.

Problem 1 (theory)

Two matrices $A, B \in \mathbb{R}^{m \times m}$ are said to be similar, denoted as $A \sim B$ if there exists some invertible matrix T such that $B = T^{-1} A T$. Prove that matrix similarity is an equivalence relation.

SOLUTION. Verify equivalence relation properties for $A, B, C \in \mathbb{R}^{m \times m}$:

Reflection. $A \sim A$ holds with $T = I$, $A = I^{-1} A I$.

Symmetry. $A \sim B \Rightarrow B \sim A$. If $A \sim B$, then $\exists T$ such that $B = T^{-1} A T$. Multiply on left by T, on right by T^{-1} to obtain $A = T B T^{-1} = (T^{-1})^{-1} B (T^{-1}) = S^{-1} B S$, hence $A \sim B$.

Transitivity. $A \sim B \wedge B \sim C \Rightarrow A \sim C$. If $A \sim B \wedge B \sim C$, then $\exists T, U$ such that $B = T^{-1} A T$ and $C = U^{-1} B U$. Replace B to obtain

$$C = U^{-1} T^{-1} A T U = (T U)^{-1} A T U = S A S,$$

with $S = T U$, hence $C \sim A$.

Matrix similarity is verified to be an equivalence relation.

Problem 2 (theory)

A matrix $A \in \mathbb{R}^{m \times m}$ is said to be skew-symmetric if $A = -A^T$.

a) Are skew-symmetric matrices a vector subspace of the vector space of matrices $(\mathbb{R}^{m \times m}, \mathbb{R}, +, \cdot)$?

b) Compute $x^T A x$ when A is skew-symmetric.

SOLUTION.

a) Verify closure for $\alpha A + \beta B$, with α, β scalars, $A, B \in \mathbb{R}^{m \times m}$. Compute $(\alpha A + \beta B)^T = \alpha A^T + \beta B^T = -(\alpha A + \beta B)$, verified as skew-symmetric, hence a subspace of $(\mathbb{R}^{m \times m}, \mathbb{R}, +, \cdot)$.

b) Compute transpose $(x^T A x)^T = x^T A^T (x^T)^T = -x^T A x$. Since $\alpha = x^T A x$ is a scalar, $\alpha^T = \alpha = -\alpha$, hence $\alpha = 0$.

Problem 3 (theory)

Consider for $v \in \mathbb{R}^m$, $\|v\|_2 = 1$, the matrix $H = I - 2 v v^T$.

a) Show that H is orthogonal.

b) Can v be chosen such that H is a projector?

SOLUTION.

a) Compute

$$\boldsymbol{H}\boldsymbol{H}^T = (\boldsymbol{I} - 2\boldsymbol{v}\boldsymbol{v}^T)(\boldsymbol{I} - 2\boldsymbol{v}\boldsymbol{v}^T)^T = (\boldsymbol{I} - 2\boldsymbol{v}\boldsymbol{v}^T)(\boldsymbol{I}^T - (2\boldsymbol{v}\boldsymbol{v}^T)^T) = (\boldsymbol{I} - 2\boldsymbol{v}\boldsymbol{v}^T)(\boldsymbol{I} - 2\boldsymbol{v}\boldsymbol{v}^T) =$$

Since $\|\boldsymbol{v}\|_2 = \boldsymbol{v}^T\boldsymbol{v} = 1$, obtain

$$\boldsymbol{H}\boldsymbol{H}^T = \boldsymbol{I} - 2\boldsymbol{v}\boldsymbol{v}^T - 2\boldsymbol{v}\boldsymbol{v}^T + 4\boldsymbol{v}\,(\boldsymbol{v}^T\boldsymbol{v})\,\boldsymbol{v}^T = \boldsymbol{I} - 4\boldsymbol{v}\boldsymbol{v}^T + 4\boldsymbol{v}\boldsymbol{v}^T = \boldsymbol{I}.$$

And since $\boldsymbol{H}^T = (\boldsymbol{I} - 2\boldsymbol{v}\boldsymbol{v}^T)^T = \boldsymbol{I} - 2\boldsymbol{v}\boldsymbol{v}^T = \boldsymbol{H}$, replacing in above gives $\boldsymbol{H}^T\boldsymbol{H} = \boldsymbol{I}$, hence $\boldsymbol{H}$ is orthogonal.

b) For $\boldsymbol{H}$ to be a projector $\boldsymbol{H}^2 = \boldsymbol{H}$, must hold. Compute

$$\boldsymbol{H}^2 = \boldsymbol{H}\boldsymbol{H} = \boldsymbol{H}\boldsymbol{H}^T = \boldsymbol{I} = \boldsymbol{I} - 2\boldsymbol{v}\boldsymbol{v}^T,$$

implying $\boldsymbol{v}\boldsymbol{v}^T = \boldsymbol{0}$ or $\boldsymbol{v} = \boldsymbol{0}$, contradicting $\|\boldsymbol{v}\|_2 = 1$, hence $\boldsymbol{H}$ cannot be a projector.

Problem 4 (computation)

The matrix $\boldsymbol{A}$ contains surveying data (https://math.nist.gov/MatrixMarket/data/Harwell-Boeing/lsq/well1033.html), and its singular value decomposition is $\boldsymbol{A} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^T$.

```
octave] cd /home/student/courses/MATH547ML/data/lsq; load well1033
```
```
octave] A=Expression1;
```
```
octave]
```

a) Construct a vector $\boldsymbol{b} = \boldsymbol{U}\boldsymbol{z}$, with $\boldsymbol{z}$ a unit vector of random numbers ($\|\boldsymbol{z}\| = 1$).

b) Solve the least squares problem $\min_x \|\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}\|$.

c) What is the error $e = \|\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}\|$?

Remember to briefly comment the solution you present. Presentation of Octave commands without explanation of what you are doing does not received full credit.

SOLUTION.

a) Compute SVD and generate random numbers

```
octave] [U,S,V]=svd(A); [m,n]=size(A); z=rand(m,1); z=z/norm(z); b=U*z;
        disp([norm(z) norm(b)])
    1.0000   1.0000
```
```
octave]
```

b) The solution is the projection of $\boldsymbol{b}$ onto $C(\boldsymbol{A})$, $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{P}_{C(\boldsymbol{A})}\boldsymbol{b} = \boldsymbol{y}$. The projector can be computed from the QR-decomposition, $\boldsymbol{A} = \boldsymbol{Q}\boldsymbol{R}$ as $\boldsymbol{P}_{C(\boldsymbol{A})} = \boldsymbol{Q}\boldsymbol{Q}^T$.

```
octave] [Q,R]=qr(A); y=Q*Q'*b; x=A\y;
```
```
octave]
```

The projector can also be computed from the SVD, $\boldsymbol{P}_{C(\boldsymbol{A})} = \boldsymbol{U}_r\boldsymbol{U}_r^T$ with $\boldsymbol{U}_r$ the first r columns of $\boldsymbol{A}$, and $r = \operatorname{rank}(\boldsymbol{A})$.

```
octave] r=rank(A); u=U(:,1:r)*U(:,1:r)'*b; w=A\u; disp(norm(x-w));
    1.2416e-14
```

```
octave]
```

c) Compute error

```
octave] err=norm(A*x-b)

err =   0.81930

octave]
```

Problem 5 (computation)

For the same data as in Problem 4:

a) Construct a reduced matrix $R = C^T A B$, that contains the first k singular modes of A, such that $\sigma_k/\sigma_1 \leqslant 10^{-1}$, with B, C orthogonal matrices.

b) Find the projection v of b from problem 4 onto the column space of C, $v \in C(C)$.

c) Solve the problem $Ru = v$.

d) Compare Bu to least squares solution from Problem 4.

Remember to briefly comment the solution you present. Presentation of Octave commands without explanation of what you are doing does not received full credit.

SOLUTION.

a) Extract the singular values, check condition $\sigma_k/\sigma_1 \leqslant 10^{-1}$ to find k.

```
octave] s=diag(S); k=310; s(k)/s(1)

ans =   0.073165

octave]
```

   Define B, C from singular vectors of A

```
octave] C=U(:,1:k); B=V(:,1:k); R=C'*A*B;

octave]
```

b) Since C has orthonormal columns, the projection is solution of $P_{C(C)}b = CC^T = Cv$, or $v = C^T b$

```
octave] v=C'*b;

octave]
```

c) Find solution

```
octave] u=R\v;

octave]
```

d) Compute the relative error $\|Bu - x\|/\|x\|$

```
octave] norm(B*u-x)/norm(x)

ans =   0.99360

octave]
```

   The above relative error is large. Define a function to carry above steps for some k

```
octave] function [rerr,Aerr]=ReducedModel(k,A,x,b,s,U,V)
          C=U(:,1:k); B=V(:,1:k); R=C'*A*B;
          v=C'*b; u=R\v;
          rerr = norm(B*u-x)/norm(x);
          Aerr=norm(A-U(:,1:k)*diag(s(1:k))*V(:,1:k)');
        end
octave] [rerr,Aerr]=ReducedModel(310,A,x,b,s,U,V)
```

```
rerr =   0.99360
Aerr =   0.099048
```

```
octave] [rerr,Aerr]=ReducedModel(315,A,x,b,s,U,V)
```

```
rerr =   0.96736
Aerr =   0.028965
```

```
octave] [rerr,Aerr]=ReducedModel(319,A,x,b,s,U,V)
```

```
rerr =   0.74059
Aerr =   0.010874
```

```
octave] [rerr,Aerr]=ReducedModel(320,A,x,b,s,U,V)
```

```
rerr =      1.1217e-14
Aerr =      2.5645e-14
```

```
octave]
```

From the above, there is no benefit from model reduction in this case.