1. MATH547 HOMEWORK 0

Topic:	Math@UNC environment
Post date:	May 9, 2020
Due date:	May 13, 2020

1.1. Background

This homework is meant to familiarize yourself with basic operations within the Math@UNC environment, and is meant to be worked through in TeXmacs, a public-domain scientific editing platform. The TeXmacs website provides several tutorials. The key features of TeXmacs that motivate adoption of the platform for this course are:

- Simple, efficient editing of mathematical content. The editor has a default text mode, and also a mathematics mode triggered by inserting an equation from the menu using Insert->Mathematics->(formula type), or the keyboard through key-strokes $, or Alt-Shift-$. Here is an example: the solution of the linear system $\boldsymbol{A}\boldsymbol{x}=\boldsymbol{b}$ with a symmetric maatrix, $\boldsymbol{A}^T=\boldsymbol{A}$, can be found by gradient descent

$$\phi(\boldsymbol{x})=\frac{1}{2}\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x}-\boldsymbol{b}^T\boldsymbol{x}, \boldsymbol{x}^{(k+1)}=\boldsymbol{x}^{(k)}-\lambda\nabla\phi(\boldsymbol{x}^{(k)}).$$

$$\phi(\boldsymbol{x})=\frac{1}{2}\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x}-\boldsymbol{b}^T\boldsymbol{x}$$

- Sessions from other mathematical packages can be inserted directly into a document. Octave is used extensively in this course, and the menu item Insert->Session->Octave leads to creation of space within the document to execute octave instructions.

```
octave] A=[1 2 3; -1 0 1; 2 1 -2]; disp(A)
   1   2   3
  -1   0   1
   2   1  -2
```

```
octave] rref(A)
ans =

   1   0   0
   0   1   0
   0   0   1
```

```
octave] inv(A)
ans =

   0.25000  -1.75000  -0.50000
   0.00000   2.00000   1.00000
   0.25000  -0.75000  -0.50000
```

```
octave] A+A
ans =

   2   4   6
  -2   0   2
   4   2  -4
```

```
octave]
```

- Documents can readily be converted to other formats: PDF, LaTeX, HTML. All course documents, including the website are produced with TeXmacs.

1.2. Theoretical questions

1.2.1. Text editing in TeXmacs

Problem. Write an itemized list of ingredients in your favorite dessert recipe

Answer.

→ 1/2 cup sugar

→ 1/2 cup packed brown sugar

→ 3 tablespoons all-purpose flour

1.2.2. Inline mathematics

Problem. The fundamental theorem of calculus states $\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a)$ for $F'(x) = f(x)$. Apply this result for $a = 0$, $b = \pi$, $f(x) = \sin x$, $F(x) = -\cos x$. Write your answer inline.

Answer. $\int_0^\pi \sin(x)\,\mathrm{d}x = -\cos(\pi) + \cos(0)$

1.2.3. Displayed mathematics

Problem. A matrix is a row of column vectors, $\boldsymbol{A} = [\, \boldsymbol{a}_1 \ \ \boldsymbol{a}_2 \ \ \ldots \ \ \boldsymbol{a}_n \,] \in \mathbb{R}^{m \times n}$, which can be expressed in terms of vector components as

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{\mathrm{mn}} \end{bmatrix}.$$

Look up the definition of a Hilbert matrix $\boldsymbol{H}$ and write in the above forms, both as a row of column vectors, and as components.

Answer. A Hilbert matrix is defined with components $h_{ij} = \frac{1}{i + j - 1}$,

$$\boldsymbol{H} = [\, \boldsymbol{h}_1 \ \ \boldsymbol{h}_2 \ \ \ldots \ \ \boldsymbol{h}_m \,], \boldsymbol{H} = \begin{bmatrix} h_{11} & h_{12} & \ldots & h_{1m} \\ h_{21} & h_{22} & \ldots & h_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ h_{m1} & h_{m2} & \ldots & h_{mm} \end{bmatrix} = \begin{bmatrix} \frac{1}{1} & \frac{1}{2} & \ldots & \frac{1}{m} \\ \frac{1}{2} & \frac{1}{3} & \ldots & \frac{1}{m+1} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{m} & \frac{1}{m+1} & \ldots & \frac{1}{2m-1} \end{bmatrix}$$

1.2.4. Octave session

Problem. Insert an Octave session and use the `hilb` function to display the Hilbert matrix $\boldsymbol{H} \in \mathbb{R}^{4 \times 4}$.

Answer.

```
octave] H=hilb(4); disp(H)
   1.00000   0.50000   0.33333   0.25000
   0.50000   0.33333   0.25000   0.20000
   0.33333   0.25000   0.20000   0.16667
   0.25000   0.20000   0.16667   0.14286
octave]
```

1.2.5. Maxima session

Problem. Insert a Maxima session, and use the `integrate` function to compute the definite integral from Problem 2.1. When in the Maxima session, a menu of commonly used commands appears.

Answer.

```
;;; Loading #P"/usr/lib/ecl-16.1.3/sb-bsd-sockets.fas"
;;; Loading #P"/usr/lib/ecl-16.1.3/sockets.fas"
;;; Loading #P"/usr/lib/ecl-16.1.3/defsystem.fas"
;;; Loading #P"/usr/lib/ecl-16.1.3/cmp.fas"
```

```
(%i2) integrate(sin(x),x,0,%pi);
```

```
(%o2) 2
```

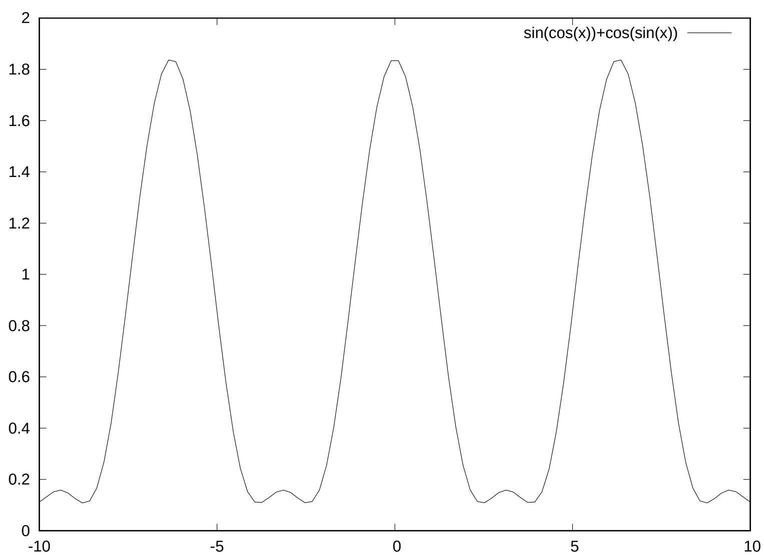
```
(%i3)
```

1.2.6. Gnuplot session

Problem. Insert a Gnuplot session to plot the function $f(x) = \sin(\cos(x)) + \cos(\sin(x))$.

Answer.

```
GNUplot] plot sin(cos(x))+cos(sin(x))
```




```
GNUplot]
```

1.3. Data Science Application

Topic: Use least squares to carry out linear regression, i.e., fitting a line to data.

1.3.1. Generate synthetic data

Problem. The following generates data by random perturbation of points on a line $y = c_0 + c_1 x$.

```
octave] m=500; x=(0:m-1)/m; c0=-1; c1=1; yex=c0+c1*x; y=(yex+rand(1,m)-0.5)';
```

```
octave] plot(x,yex,x,y,'.')
```

```
octave] m=500
```

```
m =  500
```

```
octave] x=(0:m-1)/m;
```

```
octave] x(1:4)
```

```
ans =

   0.0000000   0.0020000   0.0040000   0.0060000
```

```
octave]
```

Repeat for different values of m, c_0, c_1.

Answer.

1.3.2. Form the normal system

Problem. Define matrices $\boldsymbol{X} = [\ \mathbf{1}\ \ \boldsymbol{x}\ ]$, $\boldsymbol{N} = \boldsymbol{X}^T\boldsymbol{X}$, and vector $\boldsymbol{b} = \boldsymbol{X}^T\boldsymbol{y}$

```
octave] X=ones(m,2); X(:,2)=x(:); N=X'*X; b=X'*y;
```

```
octave] X=ones(m,2);
```

```
octave] size(X)
```

```
ans =

   500     2
```

```
octave]  X(:,2)=x(:);
```

```
octave] N=X'*X;
```

```
octave] size(N)
```

```
ans =

   2   2
```

```
octave] b=X'*y;
```

```
octave] size(b)
```

```
ans =

   2   1
```

```
octave]
```

Repeat the above, one instruction at a time, and display the first two rows of $\boldsymbol{X}, \boldsymbol{N}, \boldsymbol{b}$.

Answer.

1.3.3. Solve the least square problem

Problem. Solve the system $\boldsymbol{N}\boldsymbol{c} = \boldsymbol{b}$ by use of the Octave backslash operator `c=N\b`. Display the coefficient vector $\boldsymbol{c}$, and compare to the values you chose in Question 3.1. Also compute $\tilde{\boldsymbol{y}} = \boldsymbol{X}\boldsymbol{c}$, using `ytilde` as a notation.

Answer.

```
octave] c=N\b;
```

```
octave] size(c)
ans =

   2   1
```

```
octave] c
c =

  -1.0226
   1.0165
```

```
octave] ytilde=X*c;
```

```
octave]
```

1.3.4. Plot the result

Problem. The following Octave instructions generate an Encapsulated Postscript file showing the result of the linear regression. Familiarize yourself with the syntax and purpose of each instruction. The menu item Insert->Image->Big figure has been used to insert a figure environemnt in the Answer (Figure 1). Move your cursor inside Figure 1. Use menu item Insert->Image->Link image, to link to the Encapsulated Postscript file making the image width 4in, and leaving height empty. Edit the title of your figure.

```
octave] plot(x,y,'.k',x,ytilde,'og',x,yex,'r');
        title('Linear regression example');
        xlabel('x'); ylabel('y,yex,ytilde');
        cd /home/student/courses/MATH547ML;
        print hw00Fig01.eps;
```

```
octave]
```

Answer.

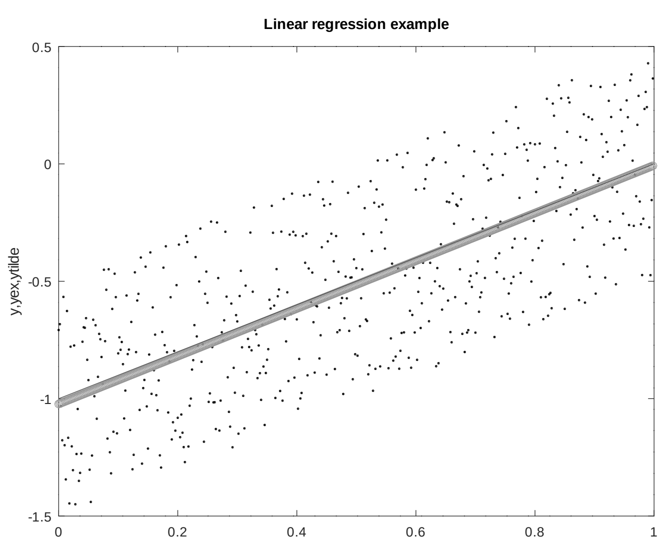


Figure 1.

Submission instructions. Save your work, and also export to PDF (menu File->Export->Pdf). In Sakai submit the files:

- hw00.tm
- any figure files you generated (e.g., that from Question 3.4)
- hw00.pdf