DATA REDUNDANCY

1. Linear dependence

For the simple scalar mapping $f:\mathbb{R}\to\mathbb{R}$, $f(x)=ax$, the condition $f(x)=0$ implies either that $a=0$ or $x=0$. Note that $a=0$ can be understood as defining a zero mapping $f(x)=0$. Linear mappings between vector spaces, $\boldsymbol{f}:U\to V$, can exhibit different behavior, and the condtion $\boldsymbol{f}(\boldsymbol{x})=\boldsymbol{A}\boldsymbol{x}=\boldsymbol{0}$, might be satisfied for both $\boldsymbol{x}\neq\boldsymbol{0}$, and $\boldsymbol{A}\neq\boldsymbol{0}$. Analogous to the scalar case, $\boldsymbol{A}=\boldsymbol{0}$ can be understood as defining a zero mapping, $\boldsymbol{f}(\boldsymbol{x})=\boldsymbol{0}$.

In vector space $\mathscr{V}=(V,S,+,\cdot)$, vectors $\boldsymbol{u},\boldsymbol{v}\in V$ related by a scaling operation, $\boldsymbol{v}=a\boldsymbol{u}$, $a\in S$, are said to be colinear, and are considered to contain redundant data. This can be restated as $\boldsymbol{v}\in\operatorname{span}\{\boldsymbol{u}\}$, from which it results that $\operatorname{span}\{\boldsymbol{u}\}=\operatorname{span}\{\boldsymbol{u},\boldsymbol{v}\}$. Colinearity can be expressed only in terms of vector scaling, but other types of redundancy arise when also considering vector addition as expressed by the span of a vector set. Assuming that $\boldsymbol{v}\notin\operatorname{span}\{\boldsymbol{u}\}$, then the strict inclusion relation $\operatorname{span}\{\boldsymbol{u}\}\subset\operatorname{span}\{\boldsymbol{u},\boldsymbol{v}\}$ holds. This strict inclusion expressed in terms of set concepts can be transcribed into an algebraic condition.

DEFINITION. *The vectors $\boldsymbol{a}_1,\boldsymbol{a}_2,\ldots,\boldsymbol{a}_n\in V$, are* linearly dependent *if there exist n scalars, $x_1,\ldots,x_n\in S$, at least one of which is different from zero such that*

$$x_1\boldsymbol{a}_1+\ldots+x_n\boldsymbol{a}_n=\boldsymbol{0}.$$

Introducing a matrix representation of the vectors

$$\boldsymbol{A}=[\ \boldsymbol{a}_1\ \ \boldsymbol{a}_2\ \ \ldots\ \ \boldsymbol{a}_n\ ];\boldsymbol{x}=\begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix}$$

allows restating linear dependence as the existence of a non-zero vector, $\exists\boldsymbol{x}\neq\boldsymbol{0}$, such that $\boldsymbol{A}\boldsymbol{x}=\boldsymbol{0}$. Linear dependence can also be written as $\boldsymbol{A}\boldsymbol{x}=\boldsymbol{0}\not\Rightarrow\boldsymbol{x}=\boldsymbol{0}$, or that one cannot deduce from the fact that the linear mapping $\boldsymbol{f}(\boldsymbol{x})=\boldsymbol{A}\boldsymbol{x}$ attains a zero value that the argument itself is zero. The converse of this statement would be that the only way to ensure $\boldsymbol{A}\boldsymbol{x}=\boldsymbol{0}$ is for $\boldsymbol{x}=\boldsymbol{0}$, or $\boldsymbol{A}\boldsymbol{x}=\boldsymbol{0}\Rightarrow\boldsymbol{x}=\boldsymbol{0}$, leading to the concept of linear independence.

DEFINITION. *The vectors $\boldsymbol{a}_1,\boldsymbol{a}_2,\ldots,\boldsymbol{a}_n\in V$, are* linearly independent *if the only n scalars, $x_1,\ldots,x_n\in S$, that satisfy*

$$x_1\boldsymbol{a}_1+\ldots+x_n\boldsymbol{a}_n=\boldsymbol{0}, \tag{1}$$

are $x_1=0$, $x_2=0$,...,$x_n=0$.

2. Basis and dimension

Vector spaces are closed under linear combination, and the span of a vector set $\mathscr{B}=\{\boldsymbol{a}_1,\boldsymbol{a}_2,\ldots\}$ defines a vector subspace. If the entire set of vectors can be obtained by a spanning set, $V=\operatorname{span}\mathscr{B}$, extending $\mathscr{B}$ by an additional element $\mathscr{C}=\mathscr{B}\cup\{\boldsymbol{b}\}$ would be redundant since $\operatorname{span}\mathscr{B}=\operatorname{span}\mathscr{C}$. This is recognized by the concept of a basis, and also allows leads to a characterization of the size of a vector space by the cardinality of a basis set.

DEFINITION. *A set of vectors $\boldsymbol{u}_1,\ldots,\boldsymbol{u}_n\in V$ is a* basis *for vector space $\mathscr{V}=(V,S,+,\cdot)$ if*

1. *$\boldsymbol{u}_1,\ldots,\boldsymbol{u}_n$ are linearly independent;*
2. *$\operatorname{span}\{\boldsymbol{u}_1,\ldots,\boldsymbol{u}_n\}=V$.*

DEFINITION. *The number of vectors $\boldsymbol{u}_1,\ldots,\boldsymbol{u}_n\in V$ within a basis is the* dimension *of the vector space $\mathscr{V}=(V,S,+,\cdot)$.*

3. Dimension of matrix spaces

The domain and co-domain of the linear mapping $\boldsymbol{f}:U\to V$, $\boldsymbol{f}(\boldsymbol{x})=\boldsymbol{A}\boldsymbol{x}$, are decomposed by the spaces associated with the matrix $\boldsymbol{A}$. When $U=\mathbb{R}^n$, $V=\mathbb{R}^m$, the following vector subspaces associated with the matrix $\boldsymbol{A}\in\mathbb{R}^{m\times n}$ have been defined:

- $C(\boldsymbol{A})$ the column space of $\boldsymbol{A}$

- $C(\boldsymbol{A}^T)$ the row space of $\boldsymbol{A}$
- $N(\boldsymbol{A})$ the null space of $\boldsymbol{A}$
- $N(\boldsymbol{A}^T)$ the left null space of $\boldsymbol{A}$, or null space of $\boldsymbol{A}^T$

DEFINITION. *The rank of a matrix $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ is the dimension of its column space and is equal to the dimension of its row space.*

DEFINITION. *The nullity of a matrix $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ is the dimension of its null space.*