DATA TRANSFORMATION

1. Gaussian elimination and row echelon reduction

Suppose now that $A\,x=b$ admits a unique solution. How to find it? We are especially interested in constructing a general procedure, that will work no matter what the size of A might be. This means we seek an *algorithm* that precisely specifies the steps that lead to the solution, and that we can program a computing device to carry out automatically. One such algorithm is *Gaussian elimination.*
Consider the system

$$\left\{\begin{array}{rcl} x_1+2x_2-x_3 &=& 2\\ 2x_1-x_2+x_3 &=& 2\\ 3x_1-x_2-x_3 &=& 1 \end{array}\right.$$

The idea is to combine equations such that we have one fewer unknown in each equation. Ask: with what number should the first equation be multiplied in order to eliminate x_1 from sum of equation 1 and equation 2? This number is called a Gaussian multiplier, and is in this case -2. Repeat the question for eliminating x_1 from third equation, with multiplier -3.

$$\left\{\begin{array}{rcl} x_1+2x_2-x_3 &=& 2\\ 2x_1-x_2+x_3 &=& 2\\ 3x_1-x_2-x_3 &=& 1 \end{array}\right. \Rightarrow \left\{\begin{array}{rcl} x_1+2x_2-x_3 &=& 2\\ -5x_2+3x_3 &=& -2\\ -7x_2+2x_3 &=& -5 \end{array}\right.$$

Now, ask: with what number should the second equation be multiplied to eliminate x_2 from sum of second and third equations. The multiplier is in this case $-7/5$.

$$\left\{\begin{array}{rcl} x_1+2x_2-x_3 &=& 2\\ -5x_2+3x_3 &=& -2\\ -7x_2+2x_3 &=& -5 \end{array}\right. \Rightarrow \left\{\begin{array}{rcl} x_1+2x_2-x_3 &=& 2\\ -5x_2+3x_3 &=& -2\\ -\frac{11}{5}x_3 &=& -\frac{11}{5} \end{array}\right.$$

Starting from the last equation we can now find $x_3=1$, replace in the second to obtain $-5x_2=-5$, hence $x_2=1$, and finally replace in the first equation to obtain $x_1=1$.
The above operations only involve coefficients. A more compact notation is therefore to work with what is known as the "bordered matrix"

$$\left(\begin{array}{cccc} 1 & 2 & -1 & 2\\ 2 & -1 & 1 & 2\\ 3 & -1 & -1 & 1 \end{array}\right) \sim \left(\begin{array}{cccc} 1 & 2 & -1 & 2\\ 0 & -5 & 3 & -2\\ 0 & -7 & 2 & -5 \end{array}\right) \sim \left(\begin{array}{cccc} 1 & 2 & -1 & 2\\ 0 & -5 & 3 & -2\\ 0 & 0 & -\frac{11}{5} & -\frac{11}{5} \end{array}\right)$$

Once the above *triangular* form has been obtain, the solution is found by back substitution, in which we seek to form the identity matrix in the first 3 columns, and the solution is obtained in the last column.

$$\left(\begin{array}{cccc} 1 & 2 & -1 & 2\\ 0 & -5 & 3 & -2\\ 0 & 0 & -\frac{11}{5} & -\frac{11}{5} \end{array}\right) \sim \left(\begin{array}{cccc} 1 & 2 & -1 & 2\\ 0 & -5 & 3 & -2\\ 0 & 0 & 1 & 1 \end{array}\right) \sim \left(\begin{array}{cccc} 1 & 0 & 0 & 1\\ 0 & 1 & 0 & 1\\ 0 & 0 & 1 & 1 \end{array}\right)$$

2. LU-factorization

- We have introduced Gaussian elimination as a procedure to solve the linear system $A\,x=b$ ("find coordinates of vector b in terms of column vectors of matrix A"), $x,b\in\mathbb{R}^m, A\in\mathbb{R}^{m\times m}$
- We now reinterpret Gaussian elimination as a sequence of matrix multiplications applied to A to obtain a simpler, upper triangular form.

2.1. Example for $m=3$

Consider the system $A\,x=b$

$$\begin{cases} x_1+2x_2-x_3 &= 2 \\ 2x_1-x_2+x_3 &= 2 \\ 3x_1-x_2-x_3 &= 1 \end{cases}$$

with

$$A=\begin{pmatrix} 1 & 2 & -1 \\ 2 & -1 & 1 \\ 3 & -1 & -1 \end{pmatrix}, b=\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$$

We ask if there is a matrix L_1 that could be multiplied with A to produce a result $L_1 A$ with zeros under the main diagonal in the first column. First, gain insight by considering multiplication by the identity matrix, which leaves A unchanged

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & -1 \\ 2 & -1 & 1 \\ 3 & -1 & -1 \end{pmatrix}=\begin{pmatrix} 1 & 2 & -1 \\ 2 & -1 & 1 \\ 3 & -1 & -1 \end{pmatrix}$$

In the first stage of Gaussian multiplication, the first line remains unchanged, so we deduce that L_1 should have the same first line as the identity matrix

$$L_1=\begin{pmatrix} 1 & 0 & 0 \\ ? & ? & ? \\ ? & ? & ? \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ ? & ? & ? \\ ? & ? & ? \end{pmatrix}\begin{pmatrix} 1 & 2 & -1 \\ 2 & -1 & 1 \\ 3 & -1 & -1 \end{pmatrix}=\begin{pmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \\ 0 & -7 & 2 \end{pmatrix}$$

Next, recall the way Gaussian multipliers were determined: find number to multiply first line so that added to second, third lines a zero is obtained. This leads to the form

$$L_1=\begin{pmatrix} 1 & 0 & 0 \\ ? & 1 & 0 \\ ? & 0 & 1 \end{pmatrix}$$

Finally, identify the missing entries with the Gaussian multipliers to determine

$$L_1=\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix}$$

Verify by carrying out the matrix multiplication

$$L_1 A=\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & -1 \\ 2 & -1 & 1 \\ 3 & -1 & -1 \end{pmatrix}=\begin{pmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \\ 0 & -7 & 2 \end{pmatrix}$$

Repeat the above reasoning to come up with a second matrix L_2 that forms a zero under the main diagonal in the second column

$$L_2=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -7/5 & 1 \end{pmatrix}$$

$$L_2 L_1 A=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -7/5 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \\ 0 & -7 & 2 \end{pmatrix}=\begin{pmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \\ 0 & 0 & -11/5 \end{pmatrix}=U$$

We have obtained a matrix with zero entries under the main diagonal (an upper triangular matrix) by a sequence of matrix multiplications.

2.2. General *m* case

From the above, we assume that we can form a sequence of multiplier matrices such that the result is an upper triangular matrix U

$$L_{m-1}\ldots L_2 L_1 A=U$$

- Recall the basic operation in row echelon reduction: constructing a linear combination of rows to form zeros beneath the main diagonal, e.g.

$$\boldsymbol{A}=\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ a_{31} & a_{32} & \dots & a_{3m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mm} \end{pmatrix} \sim \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ 0 & a_{22}-\frac{a_{21}}{a_{11}}a_{12} & \dots & a_{2m}-\frac{a_{21}}{a_{11}}a_{1m} \\ 0 & a_{32}-\frac{a_{31}}{a_{11}}a_{12} & \dots & a_{3m}-\frac{a_{31}}{a_{11}}a_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a_{m2}-\frac{a_{m1}}{a_{11}}a_{12} & \dots & a_{mm}-\frac{a_{m1}}{a_{11}}a_{1m} \end{pmatrix}$$

- This can be stated as a matrix multiplication operation, with $l_{i1}=a_{i1}/a_{11}$

$$\begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ -l_{21} & 1 & 0 & \dots & 0 \\ -l_{31} & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -l_{m1} & 0 & 0 & \dots & 1 \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ a_{31} & a_{32} & \dots & a_{3m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mm} \end{pmatrix}=\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ 0 & a_{22}-l_{21}a_{12} & \dots & a_{2m}-l_{21}a_{1m} \\ 0 & a_{32}-l_{31}a_{12} & \dots & a_{3m}-l_{31}a_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a_{m2}-l_{m1}a_{12} & \dots & a_{mm}-l_{m1}a_{1m} \end{pmatrix}$$

DEFINITION. *The matrix*

$$\boldsymbol{L}_k=\begin{pmatrix} 1 & \dots & 0 & \dots & 1 \\ 0 & \ddots & 0 & \dots & 0 \\ 0 & \dots & 1 & \dots & 0 \\ 0 & \dots & -l_{k+1,k} & \dots & 0 \\ 0 & \dots & -l_{k+2,k} & \dots & 0 \\ \vdots & \dots & \vdots & \ddots & \vdots \\ 0 & \dots & -l_{m,k} & \dots & 1 \end{pmatrix}$$

with $l_{i,k}=a_{i,k}^{(k)}/a_{k,k}^{(k)}$, *and* $\boldsymbol{A}^{(k)}=(a_{i,j}^{(k)})$ *the matrix obtained after step k of row echelon reduction (or, equivalently, Gaussian elimination) is called a Gaussian multiplier matrix.*

- For $\boldsymbol{A}\in\mathbb{R}^{m\times m}$ nonsingular, the successive steps in row echelon reduction (or Gaussian elimination) correspond to successive multiplications on the left by Gaussian multiplier matrices

$$\boldsymbol{L}_{m-1}\boldsymbol{L}_{m-2}\dots\boldsymbol{L}_2\boldsymbol{L}_1\boldsymbol{A}=\boldsymbol{U}$$

- The inverse of a Gaussian multiplier is

$$\boldsymbol{L}_k^{-1}=\begin{pmatrix} 1 & \dots & 0 & \dots & 1 \\ 0 & \ddots & 0 & \dots & 0 \\ 0 & \dots & 1 & \dots & 0 \\ 0 & \dots & l_{k+1,k} & \dots & 0 \\ 0 & \dots & l_{k+2,k} & \dots & 0 \\ \vdots & \dots & \vdots & \ddots & \vdots \\ 0 & \dots & l_{m,k} & \dots & 1 \end{pmatrix}=\boldsymbol{I}-(\boldsymbol{L}_k-\boldsymbol{I})$$

- From $(\boldsymbol{L}_{m-1}\boldsymbol{L}_{m-2}\dots\boldsymbol{L}_2\boldsymbol{L}_1)\boldsymbol{A}=\boldsymbol{U}$ obtain

$$\boldsymbol{A}=(\boldsymbol{L}_{m-1}\boldsymbol{L}_{m-2}\dots\boldsymbol{L}_2\boldsymbol{L}_1)^{-1}\boldsymbol{U}=\boldsymbol{L}_1^{-1}\boldsymbol{L}_2^{-1}\cdot\ldots\cdot\boldsymbol{L}_{m-1}^{-1}\boldsymbol{U}=\boldsymbol{L}\boldsymbol{U}$$

- Due to the simple form of $\boldsymbol{L}_k^{-1}$ the matrix $\boldsymbol{L}$ is easily obtained as

$$\boldsymbol{L}=\begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ l_{2,1} & 1 & 0 & \dots & 0 & 0 \\ l_{3,1} & l_{3,2} & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ l_{m-1,1} & l_{m-1,2} & l_{m-1,3} & \dots & 1 & 0 \\ l_{m,1} & l_{m,2} & l_{m,3} & \dots & l_{m,m-1} & 1 \end{pmatrix}$$

We will show that this indeed possible if $A\,x=b$ admits a unique solution. Furthermore, the product of lower triangular matrices is lower triangular, and the inverse of a lower triangular matrix is lower triangular (same applies for upper triangular matrices). Introduce the notation

$$L^{-1}=L_{m-1}\ldots L_2 L_1$$

and obtain

$$L^{-1}A=U$$

or

$$A=L\,U$$

The above result permits a basic insight into Gaussian elimination: the procedure depends on "factoring" the matrix A into two "simpler" matrices L,U. The idea of representing a matrix as a product of simpler matrices is fundamental to linear algebra, and we will come across it repeatedly.
For now, the factorization allows us to devise the following general approach to solving $A\,x=b$

1. Find the factorization $L\,U=A$
2. Insert the factorization into $A\,x=b$ to obtain $(L\;U)\,x=L\,(U\;x)=L\;y=b$, where the notation $y=U\;x$ has been introduced. The system

   $$L\,y=b$$

   is easy to solve by forward substitution to find y for given b
3. Finally find x by backward substitution solution of

   $$U\,x=y$$

Algorithm Gauss elimination without pivoting

for $s=1$ to $m-1$
  for $i=s+1$ to m
    $t=-a_{is}/a_{ss}$
    for $j=s+1$ to m
      $a_{ij}=a_{ij}+t\cdot a_{sj}$
    $b_i=b_i+t\cdot b_s$

for $s=m$ downto 1
  $x_s=b_s/a_{ss}$
  for $i=1$ to $s-1$
    $b_i=b_i-a_{is}\cdot x_s$

return x

Algorithm Gauss elimination with partial pivoting

$p=1{:}m$ (initialize row permutation vector)
for $s=1$ to $m-1$
  piv = abs$(a_{p(s),s})$
  for $i=s+1$ to m
    mag = abs$(a_{p(i),s})$
    if mag $>$ piv then
      $\text{piv}=\text{mag}; k=p(s); p(s)=p(i); p(i)=k$
    if piv $<\epsilon$ then break(“Singular matrix”)
    $t=-a_{p(i)s}/a_{p(s)s}$

for $j = s+1$ to m
  $a_{p(i)j} = a_{p(i)j} + t \cdot a_{p(s)j}$
for $b_{p(i)} = b_{p(i)} + t \cdot b_{p(s)}$

for $s = m$ downto 1
  $x_s = b_{p(s)} / a_{p(s)s}$
  for $i = 1$ to $s-1$
    $b_{p(i)} = b_{p(i)} - a_{p(i)s} \cdot x_s$

return x

Given $\boldsymbol{A} \in \mathbb{R}^{m \times n}$

Singular value decomposition	Gram-Schmidt	Lower-upper
Transformation of coordinates	$\boldsymbol{Ax} = \boldsymbol{b}$	
$\boldsymbol{U\Sigma V^T = A}$	$\boldsymbol{QR = A}$	$\boldsymbol{LU = A}$
$(\boldsymbol{U\Sigma V^T})\boldsymbol{x} = \boldsymbol{b} \Rightarrow \boldsymbol{Uy} = \boldsymbol{b} \Rightarrow \boldsymbol{y} = \boldsymbol{U}^T\boldsymbol{b}$	$(\boldsymbol{QR})\boldsymbol{x} = \boldsymbol{b} \Rightarrow \boldsymbol{Qy} = \boldsymbol{b}, \boldsymbol{y} = \boldsymbol{Q}^T\boldsymbol{b}$	$(\boldsymbol{LU})\boldsymbol{x} = \boldsymbol{b} \Rightarrow \boldsymbol{Ly} = \boldsymbol{b}$ (forward sub to find)$\mathbf{y}$
$\boldsymbol{\Sigma z} = \boldsymbol{y} \Rightarrow \boldsymbol{z} = \boldsymbol{\Sigma}^+\boldsymbol{y}$	$\boldsymbol{Rx} = \boldsymbol{y}$ (back sub to find $\boldsymbol{x}$)	$\boldsymbol{Ux} = \boldsymbol{y}$ (back sub to find $\boldsymbol{x}$)
$\boldsymbol{V}^T\boldsymbol{x} = \boldsymbol{z} \Rightarrow \boldsymbol{x} = \boldsymbol{Vz}$		

3. Inverse matrix

By analogy to the simple scalar equation $a\,x = b$ with solution $x = a^{-1} b$ when $a \neq 0$, we are interested in writing the solution to a linear system $A\,x = b$ as $x = A^{-1} b$ for $A \in \mathbb{R}^{m \times m}$, $x \in \mathbb{R}^m$. Recall that solving $A\,x = b = I\,b$ corresponds to expressing the vector b as a linear combination of the columns of A. This can only be done if the columns of A form a basis for $\mathbb{R}^m$, in which case we say that A is *non-singular*.

DEFINITION 1. *For matrix $A \in \mathbb{R}^{m \times m}$ non-singular the inverse matrix is denoted by A^{-1} and satisfies the properties*

$$A A^{-1} = A^{-1} A = I$$

3.1. Gauss-Jordan algorithm

Computation of the inverse A^{-1} can be carried out by repeated use of Gauss elimination. Denote the inverse by $B = A^{-1}$ for a moment and consider the inverse matrix property $A\,B = I$. Introducing the column notation for B, I leads to

$$A\left(\begin{array}{ccc} B_1 & \dots & B_m \end{array}\right) = \left(\begin{array}{ccc} e_1 & \dots & e_m \end{array}\right)$$

and identification of each column in the equality states

$$A B_k = e_k, k = 1, 2, .., m$$

with e_k the column unit vector with zero components everywhere except for a 1 in row k. To find the inverse we need to simultaneously solve the m linear systems given above.

Gauss-Jordan algorithm example. Consider

$$A = \left(\begin{array}{ccc} 1 & 2 & 3 \\ -1 & 3 & 1 \\ 2 & -1 & -2 \end{array}\right)$$

To find the inverse we solve the systems $A\,B_1 = e_1, A\,B_2 = e_2, A\,B_3 = e_3$. This can be done simultaneously by working with the matrix A bordered by I

$$(A|I) = \left(\begin{array}{cccccc} 1 & 1 & 0 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 & 1 & 0 \\ 2 & 4 & -2 & 0 & 0 & 1 \end{array}\right)$$

Carry out now operations involving linear row combinations and permutations to bring the left side to I

$$\left(\begin{array}{cccccc} 1 & 1 & 0 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 & 1 & 0 \\ 2 & 4 & -2 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{cccccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 1 & 1 & 0 \\ 0 & 2 & -2 & -2 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{cccccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 1 & 1 & 0 \\ 0 & 0 & -3 & -3 & -1 & 1 \end{array}\right) \sim$$

$$\sim \left(\begin{array}{cccccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & \frac{1}{3} & -\frac{1}{3} \end{array}\right) \sim \left(\begin{array}{cccccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & \frac{2}{3} & \frac{1}{3} \\ 0 & 0 & 1 & 1 & \frac{1}{3} & -\frac{1}{3} \end{array}\right) \sim \left(\begin{array}{cccccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{3} & \frac{1}{6} \\ 0 & 0 & 1 & 1 & \frac{1}{3} & -\frac{1}{3} \end{array}\right) \sim$$

$$\left(\begin{array}{cccccc} 1 & 0 & 0 & 1 & -\frac{1}{3} & -\frac{1}{6} \\ 0 & 1 & 0 & 0 & \frac{1}{3} & \frac{1}{6} \\ 0 & 0 & 1 & 1 & \frac{1}{3} & -\frac{1}{3} \end{array}\right)$$

to obtain

$$A^{-1} = \left(\begin{array}{ccc} 1 & -\frac{1}{3} & -\frac{1}{6} \\ 0 & \frac{1}{3} & \frac{1}{6} \\ 1 & \frac{1}{3} & -\frac{1}{3} \end{array}\right)$$

4. Determinants

- $\boldsymbol{A} \in \mathbb{R}^{m \times m}$ a square matrix, $\det(\boldsymbol{A}) \in \mathbb{R}$ is the oriented volume enclosed by the column vectors of $\boldsymbol{A}$ (a parallelipiped)
- Geometric interpretation of determinants
- Determinant calculation rules
- Algebraic definition of a determinant

DEFINITION. *The determinant of a square matrix* $\boldsymbol{A} = (\ \boldsymbol{a}_1\ \ \ldots\ \ \boldsymbol{a}_m\ ) \in \mathbb{R}^{m \times m}$ *is a real number*

$$\det(A) = \left|\begin{array}{cccc} a_{11} & a_{12} & \ldots & a_{1m} \\ a_{21} & a_{22} & \ldots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mm} \end{array}\right| \in \mathbb{R}$$

giving the (oriented) volume of the parallelepiped spanned by matrix column vectors.

- $m = 2$

$$\boldsymbol{A} = \left(\begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \end{array}\right), \det(\boldsymbol{A}) = \left|\begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \end{array}\right|$$

- $m = 3$

$$\boldsymbol{A} = \left(\begin{array}{ccc} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{array}\right), \det(\boldsymbol{A}) = \left|\begin{array}{ccc} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{array}\right|$$

- Computation of a determinant with $m = 2$

$$\left|\begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \end{array}\right| = a_{11}a_{22} - a_{12}a_{21}$$

- Computation of a determinant with $m=3$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33}+a_{21}a_{32}a_{13}+a_{31}a_{12}a_{23} - a_{13}a_{22}a_{31}-a_{23}a_{32}a_{11}-a_{33}a_{12}a_{21}$$

- Where do these determinant computation rules come from? Two viewpoints
  - *Geometric viewpoint*: determinants express parallelepiped volumes
  - *Algebraic viewpoint*: determinants are computed from all possible products that can be formed from choosing a factor from each row and each column
- $m=2$

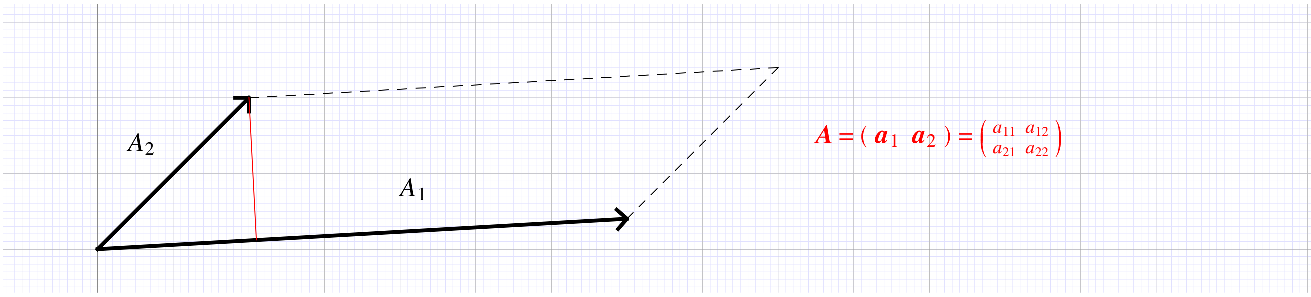

Figure 1.

- In two dimensions a ``parallelepiped'' becomes a parallelogram with area given as

$$(\text{Area}) = (\text{Length of Base}) \times (\text{Length of Height})$$

- Take $\boldsymbol{a}_1$ as the base, with length $b = \|\boldsymbol{a}_1\|$. Vector $\boldsymbol{a}_1$ is at angle φ_1 to x_1-axis, $\boldsymbol{a}_2$ is at angle φ_2 to x_2-axis, and the angle between $\boldsymbol{a}_1, \boldsymbol{a}_2$ is $\theta = \varphi_2 - \varphi_1$. The height has length

$$h = \|\boldsymbol{a}_2\| \sin\theta = \|\boldsymbol{a}_2\| \sin(\varphi_2 - \varphi_1) = \|\boldsymbol{a}_2\| (\sin\varphi_2 \cos\varphi_1 - \sin\varphi_1 \cos\varphi_2)$$

- Use $\cos\varphi_1 = a_{11} / \|\boldsymbol{a}_1\|$, $\sin\varphi_1 = a_{12} / \|\boldsymbol{a}_1\|$, $\cos\varphi_2 = a_{21} / \|\boldsymbol{a}_2\|$, $\sin\varphi_2 = a_{22} / \|\boldsymbol{a}_2\|$

$$(\text{Area}) = \|\boldsymbol{a}_1\| \, \|\boldsymbol{a}_2\| (\sin\varphi_2 \cos\varphi_1 - \sin\varphi_1 \cos\varphi_2) = a_{11}a_{22} - a_{12}a_{21}$$

- The geometric interpretation of a determinant as an oriented volume is useful in establishing rules for calculation with determinants:
  - Determinant of matrix with repeated columns is zero (since two edges of the parallelepiped are identical). Example for $m=3$

$$\Delta = \begin{vmatrix} a & a & u \\ b & b & v \\ c & c & w \end{vmatrix} = abw + bcu + cav - ubc - vca - wab = 0$$

    This is more easily seen using the column notation

$$\Delta = \det(\boldsymbol{a}_1 \; \boldsymbol{a}_1 \; \boldsymbol{a}_3 \; \ldots) = 0$$

  - Determinant of matrix with linearly dependent columns is zero (since one edge lies in the 'hyperplane' formed by all the others)
- Separating sums in a column (similar for rows)

$$\det(\boldsymbol{a}_1 + \boldsymbol{b}_1 \; \boldsymbol{a}_2 \; \ldots \; \boldsymbol{a}_m) = \det(\boldsymbol{a}_1 \; \boldsymbol{a}_2 \; \ldots \; \boldsymbol{a}_m) + \det(\boldsymbol{b}_1 \; \boldsymbol{a}_2 \; \ldots \; \boldsymbol{a}_m)$$

  with $\boldsymbol{a}_i, \boldsymbol{b}_1 \in \mathbb{R}^m$
- Scalar product in a column (similar for rows)

$$\det(\alpha \boldsymbol{a}_1 \; \boldsymbol{a}_2 \; \ldots \; \boldsymbol{a}_m) = \alpha \det(\boldsymbol{a}_1 \; \boldsymbol{a}_2 \; \ldots \; \boldsymbol{a}_m)$$

with $\alpha \in \mathbb{R}$

- Linear combinations of columns (similar for rows)

$$\det(\, \boldsymbol{a}_1 \;\; \boldsymbol{a}_2 \;\; \ldots \;\; \boldsymbol{a}_m \,) = \det(\, \boldsymbol{a}_1 \;\; \alpha \boldsymbol{a}_1 + \boldsymbol{a}_2 \;\; \ldots \;\; \boldsymbol{a}_m \,)$$

  with $\alpha \in \mathbb{R}$.

- A determinant of size m can be expressed as a sum of determinants of size $m-1$ by expansion along a row or column

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & \ldots & a_{1m} \\ a_{21} & a_{22} & a_{23} & \ldots & a_{2m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \ldots & a_{mm} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} & \ldots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m2} & a_{m3} & \ldots & a_{mm} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} & \ldots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m3} & \ldots & a_{mm} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} & a_{24} & \ldots & a_{2m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m4} & \ldots & a_{mm} \end{vmatrix} - \ldots + (-1)^{m+1} a_{1m} \begin{vmatrix} a_{21} & a_{23} & \ldots & a_{2,m-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m3} & \ldots & a_{m,m-1} \end{vmatrix}$$

- The formal definition of a determinant

$$\det A = \sum_{\sigma \in \Sigma} \nu(\sigma) a_{1i_1} a_{2i_2} \ldots a_{mi_m}$$

  requires $mm!$ operations, a number that rapidly increases with m

- A more economical determinant is to use row and column combinations to create zeros and then reduce the size of the determinant, an algorithm reminiscent of Gauss elimination for systems
  Example:

$$\begin{vmatrix} 1 & 2 & 3 \\ -1 & 0 & 1 \\ -2 & -1 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 3 & 10 \end{vmatrix} = \begin{vmatrix} 2 & 4 \\ 3 & 10 \end{vmatrix} = 20 - 12 = 8$$

  The first equality comes from linear combinations of rows, i.e. row 1 is added to row 2, and row 1 multiplied by 2 is added to row 3. These linear combinations maintain the value of the determinant. The second equality comes from expansion along the first column

4.1. Cross product

- Consider $u, v \in \mathbb{R}^3$. We've introduced the idea of a scalar product

$$u \cdot v = u^T v = u_1 v_1 + u_2 v_2 + u_3 v_3$$

  in which from two vectors one obtains a scalar

- We've also introduced the idea of an exterior product

$$u v^T = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} (\, v_1 \;\; v_2 \;\; v_3 \,) = \begin{pmatrix} u_1 v_1 & u_1 v_2 & u_1 v_3 \\ u_2 v_1 & u_2 v_2 & u_2 v_3 \\ u_3 v_1 & u_3 v_2 & u_3 v_3 \end{pmatrix}$$

  in which a matrix is obtained from two vectors

- Another product of two vectors is also useful, the cross product, most conveniently expressed in determinant-like form

$$u \times v = \begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = (u_2 v_3 - v_2 u_3)\boldsymbol{e}_1 + (u_3 v_1 - v_3 u_1)\boldsymbol{e}_2 + (u_1 v_2 - v_1 u_2)\boldsymbol{e}_3$$