## Overview

- Functions
- Measurements
- Linear mapping composition

**Definition. (Relation)** . *A relation $R$ between two sets $X$ , $Y$ is a subset of the Cartesian product $X \times Y$, $R \subseteq X \times Y$.*

Homogeneous relations $H \subseteq A \times A$ are classified according to the following criteria.

**Reflection.** Relation $H$ is reflexive if $(a, a) \in H$ for any $a \in A$. The equality relation $E \subseteq \mathbb{R} \times \mathbb{R}$ is reflexive, $\forall a \in A, a = a$, the less than relation $L \subseteq \mathbb{R} \times \mathbb{R}$ is not, $1 \in R, 1 \not< 1$.

**Symmetry.** Relation $H$ is symmetric if $(a, b) \in H$ implies that $(b, a) \in H$, $(a, b) \in H \Rightarrow (b, a) \in H$. The equality relation $E \subseteq \mathbb{R} \times \mathbb{R}$ is symmetric, $a = b \Rightarrow b = a$, the less than relation $L \subseteq \mathbb{R} \times \mathbb{R}$ is not, $a < b \not\Rightarrow b < a$.

**Anti-symmetry.** Relation $H$ is anti-symmetric if $(a, b) \in H$ for $a \neq b$, then $(b, a) \notin H$. The less than relation $L \subseteq \mathbb{R} \times \mathbb{R}$ is antisymmetric, $a < b \Rightarrow b \not< a$.

**Transitivity.** Relation $H$ is transitive if $(a, b) \in H$ and $(b, c) \in H$ implies $(a, c) \in H$. for any $a \in A$. The equality relation $E \subseteq \mathbb{R} \times \mathbb{R}$ is transitive, $a = b \wedge b = c \Rightarrow a = c$, as is the less than relation $L \subseteq \mathbb{R} \times \mathbb{R}$, $a < b \wedge b < c \Rightarrow a < c$.

**Definition. (Function)** . *A function from set $X$ to set $Y$ is a relation $F \subseteq X \times Y$, that associates to $x \in X$ a single $y \in Y$.*

**Definition. (Partition)** . *A partition of a set is a grouping of its elements into non-empty subsets such that every element is included in exactly one subset.*

- Equivalence relations partition a set into equivalence classes

**Definition. (Norm)** . *A norm on the vector space* $\mathcal{V} = (V, S, +, \cdot)$ *is a function* $\| \, \| : V \to \mathbb{R}_+$ *that for* $\boldsymbol{u}, \boldsymbol{v} \in V$, $a \in S$ *satisfies:*

1. $\|\boldsymbol{v}\| = 0 \Rightarrow \boldsymbol{v} = \boldsymbol{0}$;
2. $\| a\boldsymbol{u}\| = |a| \, \|\boldsymbol{u}\|$;
3. $\|\boldsymbol{u} + \boldsymbol{v}\| \leqslant \|\boldsymbol{u}\| + \|\boldsymbol{v}\|$.

**Definition. (Inner Product)** . *An inner product in the vector space* $\mathcal{V} = (V, \mathbb{R}, +, \cdot)$ *is a function* $s: V \times V \to \mathbb{R}$ *with properties*

***Symmetry.*** *For any* $\boldsymbol{a}, \boldsymbol{x} \in V$, $s(\boldsymbol{a}, \boldsymbol{x}) = s(\boldsymbol{x}, \boldsymbol{a})$.

***Linearity in second argument.*** *For any* $\boldsymbol{a}, \boldsymbol{x}, \boldsymbol{y} \in V$, $\alpha, \beta \in \mathbb{R}$, $s(\boldsymbol{a}, \alpha\boldsymbol{x} + \beta\boldsymbol{y}) = \alpha s(\boldsymbol{a}, \boldsymbol{x}) + \beta s(\boldsymbol{a}, \boldsymbol{y})$.

***Positive definiteness.*** *For any* $\boldsymbol{x} \in V \setminus \{\boldsymbol{0}\}$, $s(\boldsymbol{x}, \boldsymbol{x}) > 0$.

- $f: A \rightarrow B$ and $g: B \rightarrow C$, a composite function, $h = g \circ f$, $h: A \rightarrow C$ is defined by

$$h(x) = g(f(x)).$$

- $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{x}, \boldsymbol{g}(\boldsymbol{y}) = \boldsymbol{B}\boldsymbol{y}, \boldsymbol{A} \in \mathbb{R}^{m \times n}, \boldsymbol{B} \in \mathbb{R}^{p \times m}$

$$\boldsymbol{h}(\boldsymbol{x}) = \boldsymbol{C}\boldsymbol{x} = \boldsymbol{B}\boldsymbol{A}\boldsymbol{x}. \quad (1)$$

- Matrix-matrix product is a grouping of matrix-vector products

$$\boldsymbol{C} = [\ \boldsymbol{c}_1 \ \ \boldsymbol{c}_2 \ \ \ldots \ \ \boldsymbol{c}_n\ ] = [\boldsymbol{B}\ \boldsymbol{a}_1 \ \ \boldsymbol{B}\boldsymbol{a}_2 \ \ \ldots \ \ \boldsymbol{B}\boldsymbol{a}_n\ ]$$