- What is linear algebra and why is to so important to so many applications?
- Basic operations
- Factorizations
- Fundamental theorem of linear algebra
- Exposing the structure of a linear operator between different sets through the SVD
- Exposing the structure of a linear operator between the same sets through eigendecomposition

What is linear algebra, and why is it important?

- Science acquires and organizes knowledge into theories that can be verified by *quantified tests*. Mathematics furnishes the appropriate context through rigorous definition of $\mathbb{N}, \mathbb{R}, \mathbb{Q}, \mathbb{C}$.
- Most areas of science require groups of numbers to describe an observation. To organize knowledge rules on how such groups of numbers may be combined are needed. Mathematics furnishes the concept of a *vector space* $(\mathcal{S}, \mathcal{V}, +)$
  - i formal definition of a single number: scalar, $\alpha, \beta \in \mathcal{S}$
  - ii formal definition of a group of numbers: vector, $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{V}$
  - iii formal definition of a possible way to combine vectors: $\alpha \boldsymbol{u} + \beta \boldsymbol{v}$
- Algebra is concerned with precise definition of ways to combine mathematical objects, i.e., to organize more complex knowledge as a sequence of operations on simpler objects
- Linear algebra concentrates on one particular operation: the *linear combination* $\alpha \boldsymbol{u} + \beta \boldsymbol{v}$

- It turns out that a complete theory can be built around the linear combination, and this leads to the many applications linear algebra finds in all branches of knowledge.

- Group vectors as column vectors into matrices $\boldsymbol{A} = (\begin{array}{cccc} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \dots & \boldsymbol{a}_n \end{array}) \in \mathbb{R}^{m \times n}$
- Define matrix-vector multiplication to express the basic linear combination operation

$$\boldsymbol{b} = \boldsymbol{A}\boldsymbol{x} = x_1 \boldsymbol{a}_1 + \ldots + x_n \boldsymbol{a}_n$$

- Introduce a way to switch between column and row storage through the transposition operation $\boldsymbol{A}^T$. $(\boldsymbol{A} + \boldsymbol{B})^T = \boldsymbol{A}^T + \boldsymbol{B}^T$, $(\boldsymbol{A}\boldsymbol{B})^T = \boldsymbol{B}^T \boldsymbol{A}^T$
- Transform between one set of basis vectors and another $\boldsymbol{b}\boldsymbol{I} = \boldsymbol{A}\boldsymbol{x}$
- *Linear independence* establishes when a vector cannot be described as a linear combination of other vectors, i.e., if *the only way* to satisfy $x_1 \boldsymbol{a}_1 + \ldots + x_n \boldsymbol{a}_n = \boldsymbol{0}$ is for $x_1 = \ldots = x_n = 0$, then the vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$ are linearly independent
- The *span* $\langle \boldsymbol{a}_1, \ldots, \boldsymbol{a}_n \rangle = \{ \boldsymbol{b} \mid \exists \boldsymbol{x} \in \mathbb{R}^n \, \mathrm{such\, that} \, \boldsymbol{b} = x_1 \boldsymbol{a}_1 + \ldots x_n \boldsymbol{a}_n \}$ is the set of all vectors is reachable by linear combination of $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$

- The set of vectors $\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n\}$ is a *basis* of a vector space $\mathcal{V}$ if $\langle \boldsymbol{a}_1, \ldots, \boldsymbol{a}_n \rangle = \mathcal{V}$, and $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$ are linearly independent

- The number of vectors in a basis is the *dimension* of a vector space.

- Any linear operator $A: \mathcal{D} \to \mathcal{C}$, $A(\alpha \boldsymbol{u} + \beta \boldsymbol{v}) = \alpha A(\boldsymbol{u}) + \beta A(\boldsymbol{v})$ can be characterized by a matrix
- For each matrix $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ there exist four fundamental subspaces:
  1. *Column space*, $C(\boldsymbol{A}) = \{\boldsymbol{b} \in \mathbb{R}^m | \exists \boldsymbol{x} \in \mathbb{R}^n \text{ such that } \boldsymbol{b} = \boldsymbol{A}\boldsymbol{x}\} \subseteq \mathbb{R}^m$, the part of $\mathbb{R}^m$ *reachable* by linear combination of columns of $\boldsymbol{A}$
  2. *Left null space*, $N(\boldsymbol{A}^T) = \{\boldsymbol{y} \in \mathbb{R}^m | \boldsymbol{A}^T\boldsymbol{y} = 0\} \subseteq \mathbb{R}^m$, the part of $\mathbb{R}^m$ *not reachable* by linear combination of columns of $\boldsymbol{A}$
  3. *Row space*, $R(\boldsymbol{A}) = C(\boldsymbol{A}^T) = \{\boldsymbol{c} \in \mathbb{R}^n | \exists \boldsymbol{y} \in \mathbb{R}^m \text{ such that } \boldsymbol{c} = \boldsymbol{A}^T\boldsymbol{y}\} \subseteq \mathbb{R}^n$, the part of $\mathbb{R}^m$ *reachable* by linear combination of rows of $\boldsymbol{A}$
  4. *Null space*, $N(\boldsymbol{A}) = \{\boldsymbol{x} \in \mathbb{R}^n | \boldsymbol{A}\boldsymbol{x} = 0\} \subseteq \mathbb{R}^n$, the part of $\mathbb{R}^m$ *not reachable* by linear combination of rows of $\boldsymbol{A}$

The fundamental theorem of linear algebra (FTLA) states

$$C(\boldsymbol{A}),\ N(\boldsymbol{A}^T) \ \leq\ \mathbb{R}^m,$$

$$C(\boldsymbol{A}) \perp N(\boldsymbol{A}^T), \qquad C(\boldsymbol{A}) \cap N(\boldsymbol{A}^T) = \{\boldsymbol{0}\},\ C(\boldsymbol{A}) \oplus N(\boldsymbol{A}^T) = \mathbb{R}^m$$

$$C(\boldsymbol{A}^T),\quad N(\boldsymbol{A}) \quad \leq \quad \mathbb{R}^n,$$

$$C(\boldsymbol{A}^T) \perp N(\boldsymbol{A}), \qquad C(\boldsymbol{A}^T) \cap N(\boldsymbol{A}) = \{\mathbf{0}\},\quad C(\boldsymbol{A}^T) \oplus N(\boldsymbol{A}) = \mathbb{R}^n$$

- $\boldsymbol{LU} = \boldsymbol{A}$, (or $\boldsymbol{LU} = \boldsymbol{PA}$ with $\boldsymbol{P}$ a permuation matrix) Gaussian elimination, solving linear systems. Given $\boldsymbol{A} \in \mathbb{R}^{m \times m}$, $\boldsymbol{b} \in \mathbb{R}^m$, $\boldsymbol{b} \in C(\boldsymbol{A})$, find $\boldsymbol{x} \in \mathbb{R}^m$ such that $\boldsymbol{Ax} = \boldsymbol{b} = \boldsymbol{Ib}$ by:

  1. Factorize, $\boldsymbol{LU} = \boldsymbol{PA}$
  2. Solve lower triangular system $\boldsymbol{Ly} = \boldsymbol{Pb}$ by forward substitution
  3. Solve upper triangular system $\boldsymbol{Ux} = \boldsymbol{y}$ by backward substitution

- $\boldsymbol{QR} = \boldsymbol{A}$, (or $\boldsymbol{QR} = \boldsymbol{PA}$ with $\boldsymbol{P}$ a permutation matrix) Gram-Schmidt, solving least squares problem. Given $\boldsymbol{A} \in \mathbb{R}^{m \times n}$, $\boldsymbol{b} \in \mathbb{R}^m$, $n \leqslant m$, solve $\min_{\boldsymbol{x} \in \mathbb{R}^n} \|\boldsymbol{b} - \boldsymbol{Ax}\|$ by:

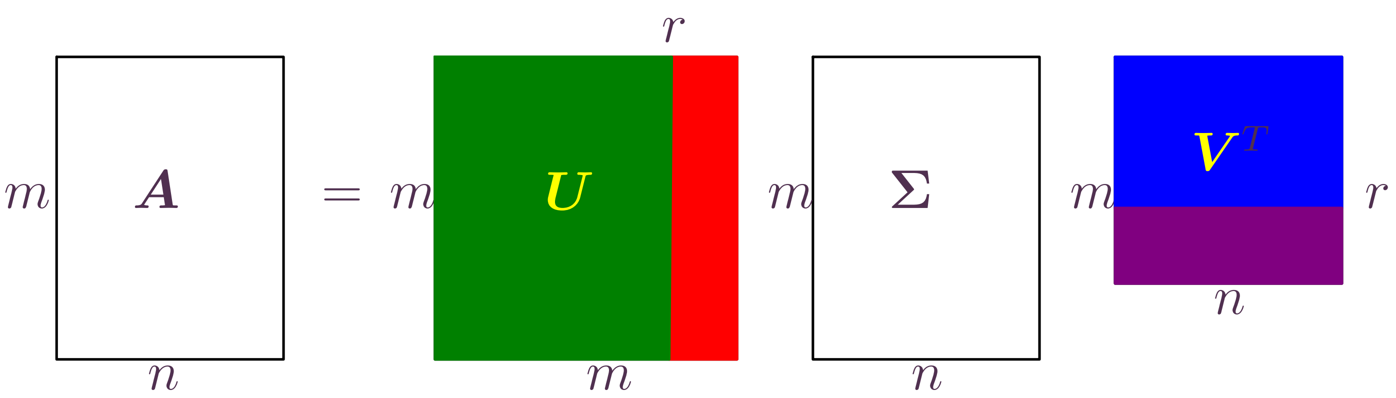
  1. Factorize, $\boldsymbol{QR} = \boldsymbol{PA}$
  2. Solve upper triangular system $\boldsymbol{Rx} = \boldsymbol{Q}^T\boldsymbol{b}$ by forward substitution

- $\boldsymbol{X\Lambda X}^{-1} = \boldsymbol{A}$, eigendecomposition of $\boldsymbol{A} \in \mathbb{R}^{m \times m}$ ($\boldsymbol{X}$ invertible if $\boldsymbol{A}$ is normal, i.e., $\boldsymbol{AA}^T = \boldsymbol{A}^T\boldsymbol{A}$)

- $\boldsymbol{QTQ}^T = \boldsymbol{A}$, Schur decomposition of $\boldsymbol{A} \in \mathbb{R}^{m \times m}$, $\boldsymbol{Q}$ orthogonal matrix, $\boldsymbol{T}$ triangular matrix, decomposition always exists
- $\boldsymbol{U \Sigma V}^T = \boldsymbol{A}$, Singular value decomposition of $\boldsymbol{A} \in \mathbb{R}^{m \times n}$, $\boldsymbol{U} \in \mathbb{R}^{m \times m}$, $\boldsymbol{V} \in \mathbb{R}^{n \times n}$ orthogonal matrices, $\boldsymbol{\Sigma} = \mathrm{diag}(\sigma_1, \sigma_2, \ldots) \in \mathbb{R}_+^{m \times n}$, decomposition always exists

SVD

- The SVD of $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ reveals: $\operatorname{rank}(\boldsymbol{A})$, bases for $C(\boldsymbol{A})$, $N(\boldsymbol{A}^T)$, $C(\boldsymbol{A}^T)$, $N(\boldsymbol{A})$



$$\boldsymbol{A} = \begin{pmatrix} \boldsymbol{u}_1 & \dots & \boldsymbol{u}_r & \boldsymbol{u}_{r+1} & \dots & \boldsymbol{u}_m \end{pmatrix} \begin{pmatrix} \sigma_1 & & & & \\ & \ddots & & & \\ & & \sigma_r & & \\ & & & 0 & \\ & & & & \ddots \end{pmatrix} \begin{pmatrix} \boldsymbol{v}_1^T \\ \vdots \\ \boldsymbol{v}_r^T \\ \boldsymbol{v}_{r+1}^T \\ \vdots \\ \boldsymbol{v}_n^T \end{pmatrix}$$